In an analysis of the performance of low-density parity-check (LDPC) codes [4] under an iterative decoding algorithm based on Belief Propagation (BP), it is important to investigate both the capability of error correction of the employed LDPC codes itself, that is, the error performance of LDPC codes under the maximum-likelihood (ML) decoding [7], and the compatibility of the employed LDPC codes with BP-based decoding [9], [10].

As for the analysis of the error performance of LDPC codes under ML decoding, it is known [7] that the average weight distribution of a given ensemble of linear codes can provide both of upper and lower bounds of the averaged ML decoding error probability. Moreover, the weight distribution of LDPC codes have been studied for various ensembles, for example, Gallager’s ensembles [4], regular ensembles [1], [5], standard irregular ensembles [1], [6] and irregular repeat-accumulate code ensembles [3].

In an analysis of the behavior of BP-based decoding over a binary erasure channel (BEC), a notion of stopping sets and the distribution of their size play important roles [1], [2], [8]. More concretely, both of block and bit errors depend on the size of the maximal stopping set in the set of variable nodes corresponding to symbols received as erasure. As for the conventional analysis on the stopping set distribution, Orlitsky et al. [8] explicitly formulate the asymptotic average block error probability of regular and standard irregular LDPC code ensembles.

All code ensembles analyzed in works introduced above are the instances of one-edge type LDPC code ensembles which are special cases of multi-edge type LDPC code ensembles proposed by Richardson et al. [9], and they have found concrete examples in their ensembles which exhibit better performance than conventional ones under BP-based decoding. Our motivation in this paper is to extend some part of conventional results, that is, to formulate the average weight and the stopping set distributions and their asymptotic exponents for two instances of two-edge type LDPC code ensembles. Moreover, by using the derived expressions, we investigate some interesting characteristics such as the symmetry and the conditions for zero of the weight distributions of both two code ensembles, and the relation between two code ensembles from the perspectives of the weight and the stopping set distributions.

This paper is organized as follows. In Sect. 2, we give definitions of ensembles of bipartite graphs investigated in this paper and the average weight and the stopping set distributions. Main results and their proofs are provided in Sect. 3. In Sect. 4, we investigate some additional characteristics. Section 5 concludes this paper. Some proofs are provided in Appendix.

2. Preliminaries

In this paper, we denote by \( \mathbb{N} \), \( \mathbb{Z} \) and \( \mathbb{Q} \) sets of natural numbers, integers and rational numbers, respectively.

2.1 Bipartite Graphs and Codes

Let \( V \) and \( C \) be sets of vertices with \( V \cap C = \emptyset \), and assume that all vertices in \( V \) are numbered from 1 to \( |V| \) and all vertices in \( C \) are numbered from 1 to \( |C| \) by some arbitrary ordering. We denote by \( G = (V \cup C, E) \) a bipartite graph which consists of a vertex set \( V \cup C \) and an edge set \( E \), every edge in which has one end in \( V \) and another in \( C \). Moreover we associate a bipartite graph \( G \) with a binary \( |C| \times |V| \) matrix \( H_G = [h_{i,j}] \) where \( h_{i,j} \) is set to be 1 if there are the odd number of edges between the \( i \)-th node in \( C \) and the \( j \)-th node in \( V \) otherwise. Then a binary linear code \( C_G \) associated with a bipartite graph \( G \) is defined as \( C_G := \{ e \in \{0,1\}^{|V|} : H_G e^T = 0^T \} \) where \( 0^T \) represents all zero vector of length \(|C|\), \( v \in V \) and \( c \in C \) are called a variable node and a check node, respectively. In this paper, we
identify an ensemble of bipartite graphs with an ensemble of binary linear codes via the association described above.

2.2 Edge Types

We first introduce the notion of a socket [10]. A socket is a place stuck on a node in a bipartite graph which an edge is plugged into. This means that the number of sockets on a node is equal to its degree. A socket on a variable (resp. check) node is called a variable (resp. check) socket.

For a bipartite graph $G = (V \cup C, E)$, let $V_1$ be a proper subset of $V$ and $V_2 := V \setminus V_1$. A variable socket on a node belonging to $V_p$ ($p = 1, 2$) is called a type-$p$ variable socket, and a check socket which is connected with a type-$p$ variable socket is called a type-$p$ check socket. The number of type-$p$ sockets on a node is said to be the type-$p$ degree. We denote by $M_p$ the number of type-$p$ variable sockets of $G$. Since the number of type-$p$ check sockets of $G$ agrees with $M_p$, both of $M_p$ type-$p$ variable and check sockets of $G$ can be numbered from 1 to $M_p$. An edge is said to be a type-$p$ edge if it is specified by a pair of a type-$p$ variable socket and a type-$p$ check socket.

2.3 Code Ensembles

In this paper, we exclusively treat two code ensembles which are denoted by $G_1(n, j_1, k_1)$ and $G_2(n, j_1, j_2, k_2)$ where $n, j_p$ and $k_p$ ($p = 1, 2$) are parameters of those ensembles.

2.3.1 Definition of $G_1(n, j_1, k_1)$

Let $n, j_1$ and $k_1$ be three positive integers. The type-1 (resp. type-2) degree of each variable node belonging to $V_1$ (resp. $V_2$) is identical, say $j_1$ (resp. 2). Also the type-1 (resp. type-2) degree of each check node is $k_1$ (resp. 2). Let $n_1 := |V_1|, n_2 := |V_2|$ and $m := |C|$. Then there are $j_1 n_1$ type-1 variable sockets and $2 n_2$ type-2 variable sockets on $V_1$ and $V_2$, respectively, and there are $k_1 m$ type-1 and $2 m$ type-2 check sockets on $C$, respectively. Thus $M_1 = j_1 n_1 = k_1 m$ and $M_2 = 2 m$.

The arrangement of edges between $V_1$ and $C$ is specified by choosing a permutation $\pi_1$ from $P(M_1)$. For each $i \in \{1, 2, \cdots, M_1\}$, $i$-th type-1 variable sockets on $V_1$ is connected with $\pi_1(i)$-th type-1 check socket on $C$. Similarly, the arrangement of edges between $V_2$ and $C$ is specified by $\pi_2 \in P(M_2)$. Then we denote by $G(n, \pi_1, \pi_2)$ a bipartite graph with $n$ variable nodes (i.e. $n = n_1 + n_2$) such that the arrangement of edges between $V_1$ and $C$ and between $V_2$ and $C$ are specified by $\pi_1 \in P(M_1)$ and $\pi_2 \in P(M_2)$, respectively.

Now, $G_1(n, j_1, k_1)$ is defined as a collection of bipartite graphs $G(n, \pi_1, \pi_2)$ where $\pi_1$ is uniformly chosen from $P(M_1)$ and $\pi_2$ is a fixed special permutation of $P(M_2)$ such that nodes of $V_2$ and $C$ and edges between $V_2$ and $C$ compose a cycle of length $M_2$.

Figure 1 shows a bipartite graph belonging to the ensemble $G_1(n, j_1, k_1)$. In Fig. 1, an open circle and an open square are a variable node and a check node, respectively.

It is easy to verify that $G_1(n, j_1, k_1)$ has following properties.

\begin{lemma}
For $(V_1 \cup V_2 \cup C, E) \in G_1(n, j_1, k_1)$, we have $n_1 = \frac{j_1}{j_1+k_1} n, n_2 = \frac{k_1}{j_1+k_1} n$ and $G_1(n, j_1, k_1) = (j_1 n_1)!$, where $n_1 = |V_1|, n_2 = |V_2|$ and $m = |C|$.
\end{lemma}

\begin{proof}
\end{proof}

2.3.2 Definition of $G_2(n, j_1, j_2, k_2)$

Let $n, j_1, j_2, k_1$ and $k_2$ be five positive integers. The type-1 (resp. type-2) degree of each variable node belonging to $V_1$ (resp. $V_2$) is $j_1$ (resp. $j_2$). Also the type-1 (resp. type-2) degree of each check node is $k_1$ (resp. $k_2$). Let $n_1 := |V_1|, n_2 := |V_2|$ and $m := |C|$. Then there are $j_1 n_1$ type-1 variable sockets and $j_2 n_2$ type-2 variable sockets on $V_1$ and $V_2$, respectively, and there are $k_1 m$ type-1 and $k_2 m$ type-2 check sockets on $C$, respectively. Thus $M_1 = j_1 n_1 = k_1 m$ and $M_2 = j_2 n_2 = k_2 m$.

As we have explained in the definition of $G_1(n, j_1, k_1)$, the arrangement of edges between $V_1$ and $C$ and between $V_2$ and $C$ are determined by choosing permutations $\pi_1 \in P(M_1)$ and $\pi_2 \in P(M_2)$, respectively.

Then $G_2(n, j_1, j_2, k_2)$ is defined as a collection of bipartite graphs $G(n, \pi_1, \pi_2)$ where $\pi_1$ and $\pi_2$ are uniformly chosen from $P(M_1)$ and $P(M_2)$, respectively. Figure 2 shows a bipartite graph belonging to the ensemble $G_2(n, j_1, j_2, k_2)$.

It is easy to verify that $G_2(n, j_1, j_2, k_2)$ has following properties.

\begin{lemma}
For $(V_1 \cup V_2 \cup C, E) \in G_2(n, j_1, j_2, k_2)$, we have $n_1 = \frac{j_1}{j_1+k_1+j_2} n, n_2 = \frac{j_2}{j_1+k_1+j_2} n$ and $G_2(n, j_1, j_2, k_2) = (j_1 n_1)(j_2 n_2)!$, where $n_1 = |V_1|, n_2 = |V_2|$ and $m = |C|$.
\end{lemma}

\begin{proof}
\end{proof}

2.4 Average Weight and Stopping Set Distributions

\begin{definition}
[2] Let $G = (V \cup C, E)$ be a bipartite graph.
\end{definition}
Then $V' \subset V$ is said to be a **stopping set** if no check node is connected to $V'$ by a single edge.

In Fig. 3, we illustrate a bipartite graph with six variable nodes (circles) and four check nodes (squares). $(v_1, v_2, e_4)$ is an example of stopping set, contained in this bipartite graph.

Let $G = (V \cup C, E)$ be a bipartite graph. For $C' \subset C$ and $V' \subset V$, $(V', C')$ is said to have the **odd or even connection** if every node of $C'$ is connected with odd or even number of variable nodes belonging to $V'$, respectively. Then it is easy to verify that $x \in C_G$ if and only if $(V_x, C)$ has the even connection where $V_x$ represents a set of variable nodes corresponding to $x_i = 1$. Further for a non-negative integer $i$, $V_1^i \subseteq C'$, $V_2^i \subseteq C'$ and $V_2^{i+1} \subseteq C'$ respectively mean that every node of $C'$ is connected with $i$, more than or equal to $i$, and more than or less than $i$ variable nodes belonging to $V'$. Then it is easy to verify that $V'$ is a stopping set in $G$ if and only if $V' \cong C$.

The weight distribution averaged on a given code ensemble (shortly, the average weight distribution), the distribution of the size of stopping sets averaged on a given code ensemble (shortly, the average stopping set distribution), and the asymptotic representations of those distributions are respectively defined as follows.

**Definition 2:** Let $G(n)$ be an ensemble of bipartite graphs with $n$ variable nodes and we assign uniform probability to each element of $G(n)$. Moreover, for a positive integer $n_1$ such that $1 \leq n_1 \leq n$ and each $G = (V \cup C, E) \in G(n)$, $V_1 := \{v_1, v_2, \ldots, v_{n_1}\}$ and $V_2 := V \setminus V_1$. Let $n_2 := |V_2| = n - n_1$. Then the average weight and the stopping set distributions of $G(n)$, both of which are denoted by $\overline{A}_{w_1, w_2} : w_p = 0, 1, 2, \ldots, n_p$, $(p = 1, 2)$ as long as there is no fear of confusion, are defined as

$$\overline{A}_{w_1, w_2} := \frac{\sum_{G \in G(n)} A_{w_1, w_2}(G)}{|G(n)|}.$$ (1)

For the weight distribution, $A_{w_1, w_2}(G)$ in Eq. (1) is defined as

$$A_{w_1, w_2}(G) := \{(x_1, x_2) \in C_G : x_p \in \{0, 1\}^{n_p},$$

$$w_t(x_p) = w_p \ (p = 1, 2)\}$$

where $x_p$ represents the vector of length $n_p$ corresponding to $V_p$ $(p = 1, 2)$ and $(x_1, x_2)$ denotes the concatenation of $x_1$ and $x_2$. Moreover, $w_t(x)$ represents the Hamming weight of a vector $x$. For the stopping set distribution, $A_{w_1, w_2}(G)$ in Eq. (1) is defined as

$$A_{w_1, w_2}(G) := |V'_p \cup V'_2 \subset V :$$

$$V'_p \subset V_p, |V'_p| = w_p \ (p = 1, 2),$$

$$V'_1 \cup V'_2 \text{ is a stopping set in } G||.$$ (2)

According to Definition 2, $\overline{A}_{w_1, w_2}$ for weight distribution means the average number of codewords $x = (x_1, x_2)$ among all codewords with Hamming weight $w_1 + w_2$, where $(x_1, x_2)$ represents the concatenation of two vectors $x_1$ and $x_2$ with length $n_p := |V_p|$ and $w_t(x_p) = w_p \ (p = 1, 2)$. Similarly, $\overline{A}_{w_1, w_2}$ for stopping set distribution means the average number of stopping sets $V' = V'_1 \cup V'_2$ among all stopping sets of size $w_1 + w_2$, where $V'_p$ represents a subset of $V_p$ of size $w_p \ (p = 1, 2)$.

In the case of the weight distribution, the numerator in the right hand side of Eq. (1) is transformed into

$$\sum_{G \in G(n)} A_{w_1, w_2}(G)$$

$$= \sum_{G \in G(n)} \left| \{(x_1, x_2) \in C_G : x_p \in \{0, 1\}^{n_p},\right.$$  

$$w_t(x_p) = w_p \ (p = 1, 2)\}$$

$$= \sum_{x_1 \in \{0, 1\}^{n_1}} \sum_{x_2 \in \{0, 1\}^{n_2}} \left| \{G \in G(n) : (x_1, x_2) \in C_G\} \right|$$

$$= \sum_{V'_1, V'_2} \sum_{|V'_1| = w_1 \ \text{and} \ |V'_2| = w_2} Z(V'_1, V'_2)$$ (2)

where

$$Z(V'_1, V'_2) := \left| \{G \in G(n) : (V'_1 \cup V'_2, C) \text{ has the even connection}\} \right|.$$ (3)
Similarly, in the case of the stopping set distribution, we have Eq. (2) by replacing the definition of \( Z(V'_1, V'_2) \) given in Eq. (3) with

\[
Z(V'_1, V'_2) := \left| \left\{ G \in \mathcal{G}(n) : V'_1 \cup V'_2 \supseteq \mathbb{Z}^2 \right\} \right|.
\]

Therefore in order to formulate the average weight and the stopping set distributions of \( \mathcal{G}_1(n, j_1, k_1) \) and \( \mathcal{G}_2(n, j_1, k_1, j_2, k_2) \), it is sufficient to formulate \( Z(V'_1, V'_2) \) given in Eq. (3) or Eq. (4) because \(|G_1(n, j_1, k_1)|\) and \(|\mathcal{G}_2(n, j_1, k_1, j_2, k_2)|\) have been already given in Lemma 1 and 2.

In general, for an ensemble \( \mathcal{G}(n) \) of bipartite graphs, the usual average weight and the stopping set distributions, denoted by \( \bar{A}_w := \{ w = 0, 1, \ldots, n \} \), are defined as

\[
\bar{A}_w := \frac{1}{|\mathcal{G}(n)|} \sum_{G \in \mathcal{G}(n)} A_w(G)
\]

where

\[
A_w(G) := \begin{cases} |x \in C_G : \text{wt}(x) = w|, & \text{(for the weight distribution)}, \\ |V' \subset V : |V'| = w, V' \text{ is a stopping set in } G|, & \text{(for the stopping set distribution)}. \end{cases}
\]

Then it can be easily shown that for \( w = 0, 1, \ldots, n \),

\[
\bar{A}_w = \sum_{k_1, k_2} \bar{A}_{w_1, a_2}(G)
\]

holds since \( A_w(G) = \sum_{k_1, k_2 \geq 0} A_{w_1, a_2}(G) \).

### 2.5 The Asymptotic Exponents of Weight and Stopping Set Distributions

In order to investigate the asymptotic behavior of the weight and stopping set distributions, we define the asymptotic exponents of the average weight and the stopping set distributions as

\[
a_{a_1, a_2} := \lim_{n \to \infty} \frac{1}{n} \log_2 \bar{A}_{a_1, a_2}(n)
\]

where \( a_1 := w_1/n \) and \( a_2 := w_2/n \).

For the usual average weight and the stopping set distributions \( \bar{A}_w \), we define their asymptotic exponents as

\[
a_{w} := \lim_{n \to \infty} \frac{1}{n} \log_2 \bar{A}_w(n)
\]

Then it is easily verified that for \( 0 \leq \alpha \leq 1 \),

\[
a_{w} = \max_{a_1, a_2} a_{a_1, a_2}.
\]

For a polynomial \( p(x) \), we denote by \( \text{coef}(p(x), x^\theta) \) the coefficient of \( x^\theta \) in \( p(x) \). Moreover, for a polynomial \( p(x) \), we also denote by \( \text{maxdeg}(p(x)) \) and \( \text{mindeg}(p(x)) \) the maximum and the minimum degree of term appearing in \( p(x) \), respectively.

In the estimation of the right hand side of Eq. (5), the following lemma plays a principal role.

**Lemma 3:** Let \( \mathcal{G}(n) \) be a function such that \( p(x)^n \) forms a polynomial with non-negative coefficients. Also for a positive rational number \( \delta \), let \( e_i \) be the \( i \)-th smallest element in the following set:

\[
\{ e \in \mathbb{Z} : e \geq 0, e/\eta \in \mathbb{Z}, \text{coef}(p(x)^e, x^\theta) \neq 0 \}.
\]

Then if \( \min\text{deg}(p(x)^\eta) < \eta \delta < \max\text{deg}(p(x)^\eta) \), then

\[
\lim_{i \to \infty} \frac{1}{e_i} \log_2 \text{coef}(p(x)^{e_i}, x^{\delta \theta}) = \log_2 \frac{p(x)\theta}{x_0} \]

where \( x_0 \) is a unique positive solution of \( \frac{p(x)\theta}{x} = \delta \).}

## 3. Main Results

In this section, we derive the average weight and the stopping set distributions and their asymptotic exponents of \( \mathcal{G}_1(n, j_1, k_1) \) and \( \mathcal{G}_2(n, j_1, k_1, j_2, k_2) \).

For a natural number \( n \) and \( \theta \in [0, 1] \),

\[
H_2(\theta) := \lim_{n \to \infty} \frac{1}{n} \log_2 \frac{\theta^n}{(1-\theta)^n}
= -\theta \log_2 \theta - (1-\theta) \log_2 (1-\theta)
\]

and for a natural number \( n \) and \( \theta, \theta' \in [0, 1] \) such that \( \theta + \theta' \leq 1 \),

\[
H_2(\theta, \theta') := \lim_{n \to \infty} \frac{1}{n} \log_2 \left( \frac{\theta^n \theta'^n (1-\theta - \theta')}{(1-\theta - \theta')n} \right)
= -\theta \log_2 \theta - \theta' \log_2 \theta' - (1-\theta - \theta') \log_2 (1-\theta - \theta').
\]

### 3.1 Weight Distribution of \( \mathcal{G}_1(n, j_1, k_1) \)

**Theorem 1:** For \( \mathcal{G}_1(n, j_1, k_1) \),

\[
\bar{\lambda}_{w_1, w_2} = \sum_{(j_1, m, \ell) \in \lambda_{w_1, w_2}} \sum_{\ell \in \ell_{w_1, w_2}} T(m, w_2, \ell) \cdot \text{coef} (B_{j_1}^0(x)^{2\ell} B_{j_1}^0(x)^{m-2\ell}, x^{w_1})
\]

where \( \ell_{w_1, w_2}(m, w_2) \) and \( \ell_{M}(w_1, w_2) \) are defined as

\[
\ell_{w_1, w_2} := \begin{cases} \max \left\{ \ell_{[w_1/2]}, m - \ell_{[w_2/2]} \right\}, & \left( k_1: \text{even} \right), \\ \min \left\{ \ell_{[w_1/2]}, m - \ell_{[w_2/2]} \right\}, & \left( k_1: \text{odd} \right) \end{cases}
\]

\[
\ell_{M}(w_1, w_2) := \begin{cases} \min \left\{ \ell_{[w_1/2]}, m - \ell_{[w_2/2]} \right\}, & \left( k_1: \text{even} \right), \\ \max \left\{ \ell_{[w_1/2]}, m - \ell_{[w_2/2]} \right\}, & \left( k_1: \text{odd} \right) \end{cases}
\]
solution of

**3.2 Proof of Eq. (7)**

For \(V'_1 \subset V_1\) with \(|V'_1| = w_1\), let

\[
Y_{wd}(V'_1) := \sum_{V'_2 \subset V_2 | |V'_2| = w_2} Z(V'_1, V'_2)
\]

where \(Z(V'_1, V'_2)\) is defined in Eq. (3) for \(G_1(n, j_1, k_1)\). Let \(V_2 := \{v_{2, 1}, \ldots, v_{2, m}\}\). For a non-empty proper subset \(V'_2\) of \(V_2\), \(V'_2\) is said to be divided into \(\ell\) sets if there exists a positive integer \(\ell\) such that \(V'_2 = \left( \biguplus_{j=1}^{\ell} S_j \right) \uplus U\) where \(\biguplus\) represents the disjoint union of sets. Moreover, \(S_j (i = 1, 2, \ldots, \ell)\) and \(U\) are defined as follows:

**Case 1:** If \(v_{2, 1} \in V'_2\), then \(S_i := \{v_{2, n}, v_{2, n+1}, \ldots, v_{2, h}\}\) \((i = 1, 2, \ldots, \ell)\) where for \(i = 1, 2, \ldots, \ell, a_i\) represents the smallest integer such that \(v_{2, a_i} \in V'_2 \setminus \biguplus_{j=1}^{i-1} S_j\) and \(b_i\) represents an integer such that \(v_{2, b_i+1} \not\in V'_2\) and \(v_{2, j} \in V'_2 \setminus \biguplus_{j=1}^{i-1} S_j\) for all \(j \in Z\) such that \(a_i \leq j \leq b_i\). Moreover, \(U := \{v_{2, h+1}, v_{2, h+2}, \ldots, v_{2, m}\}\) where \(a_{\ell+1}\) represents the smallest integer such that \(v_{2, a_{\ell+1}} \in V'_2 \setminus \biguplus_{j=1}^{\ell} S_j\) if \(v_{2, m} \in V'_2\) and \(U := \emptyset\) otherwise.

**Case 2:** If \(v_{2, 1} \notin V'_2\), then \(S_i := \{v_{2, n}, v_{2, n+1}, \ldots, v_{2, h}\}\) \((i = 1, 2, \ldots, \ell)\) and \(U := \emptyset\) where \(a_i\) and \(b_i\) are defined in Case 1.

For \(V'_2 = \emptyset\) and \(V'_2 = V_2\), \(V'_2\) is said to be divided into 0 set.

**Example 1:** For \(V_2 = \{v_{2, 1}, v_{2, 2}, \ldots, v_{2, 10}\}\) and \(V'_2 = \{v_{2, 3}, v_{2, 4}, v_{2, 5}, v_{2, 7}, v_{2, 8}\}\), we have \(S_1 = \{v_{2, 3}, v_{2, 4}, v_{2, 5}\}\), \(S_2 = \{v_{2, 7}, v_{2, 8}\}\), \(U = \emptyset\), and therefore \(V'_2\) is divided into two sets. □

Let \(X_{w_1, \ell} := \{V'_2 \subset V_2 : |V'_2| = w_2\}\), and for a non-negative integer \(\ell\), let \(X_{w_1, \ell} \uplus \) denote a collection of \(V'_2 \in X_{w_1, \ell}\) divided into \(\ell\) sets. Then we have the following lemma. The proof of this lemma is given in Appendix A.

**Lemma 4:** (1) \(|X_{w_1, \ell}| = T(m, w_2, \ell)\) where a function \(T\) is defined in Eq. (10).

(2) \(X_{w_1, \ell} \uplus \) represents the disjoint union of sets. □

For \(V'_2 \in X_{w_1, \ell}\), we consider the arrangement of edges between \(V'_2\) and its neighbor check nodes. Let \(r_{odd}\) and \(r_{even}\) denote the number of check nodes which are connected with \(V'_2\) by odd and even number of edges, respectively. Then we have the following lemma. The proof of this lemma is given in Appendix B.

**Lemma 5:** Let \(V'_1 \subset V_1\) with \(|V'_1| = w_1\) and \(V'_2 \in X_{w_1, \ell}\). Then

\[Z(V'_1, V'_2) = (j_1w_1)(j_1(n_1 - w_1))! \cdot \text{coef} \left( B^0_1(x)^{2\ell} B^0_2(x)^{m-2\ell}, x^{j_1w_1} \right).\]  \(16\)

\[Y_{wd}(V'_1) = (j_1w_1)(j_1(n_1 - w_1))! \sum_{\ell = \ell_{wd}(w_1, w_2)} T(m, w_2, \ell) \cdot \text{coef} \left( B^0_1(x)^{2\ell} B^0_2(x)^{m-2\ell}, x^{j_1w_1} \right). \]

\[17\]

**Proof of Eq. (7)** By substituting Eq. (17) into Eq. (2) and

\[1\] For \(i = 0, \biguplus_{j=1}^{\ell} S_j\) is defined as an empty set.
noting that \( Y_{w_2}(V'_1) \) does not depend on a choice of \( V'_1 \subset V_1 \) with \( |V'_1| = w_1 \), we have

\[
\sum_{G \in \mathcal{G}_1, (n, j_1, k_1)} \sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0 \nn \sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0 \nn \sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0
\]

\[
\sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0 \nn \sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0 \nn \sum_{\ell = \ell_0, \omega_1} \ldots \ldots \ldots \ll 0
\]

\[
\ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0
\]

\[
\ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0
\]

\[
\ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0 \nn \ldots \ldots \ldots \ll 0
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Moreover,

\[ a_{\alpha_1, \alpha_2} = \hat{\gamma}_1 (1 - j) H_2 (\frac{\alpha_1}{\hat{\gamma}_1}) + \hat{\gamma}_2 (1 - j) H_2 (\frac{\alpha_2}{\hat{\gamma}_2}) + \max \beta \left\{ \hat{R} H_2 (\frac{\beta}{\hat{R}}) + \sum_{p=1}^2 \log_{\beta} B_{\beta p}^0 (x_p) B_{\beta p}^c (x_p) \right\} \]

(22)

where \( \hat{\gamma}_1 := \frac{j_{k_1}}{j_{k_1 + k_2}}, \ \hat{\gamma}_2 := \frac{j_{k_2}}{j_{k_1 + k_2}}, \ \hat{R} := \frac{j_{k_1 k_2}}{j_{k_1 + k_2}}, \) and the maximum is taken over \( \beta \in \mathbb{Q} \) satisfying \( \beta \leq \hat{\beta}_M (\alpha_1, \alpha_2) \) where

\[ \hat{\beta}_M (\alpha_1, \alpha_2) := \begin{cases} 0, & (k_1 : \text{even}, k_2 : \text{even}), \\
\max \{0, j_2 (\alpha_2 - \frac{k_2 - 1}{k_2})\}, & (k_1 : \text{even}, k_2 : \text{odd}), \\
\max \{0, j_1 (\alpha_1 - \frac{k_1 - 1}{k_2}), j_2 (\alpha_2 - \frac{k_2 - 1}{k_2})\}, & (k_1 : \text{odd}, k_2 : \text{even}), \\
\max \{0, j_1 (\alpha_1 - \frac{k_1 - 1}{k_1}), j_2 (\alpha_2 - \frac{k_2 - 1}{k_1})\}, & (k_1 : \text{odd}, k_2 : \text{odd}). \end{cases} \]

(23)

Further for \( p = 1, 2, x_p \) in Eq. (22) is a unique positive solution of

\[ \frac{k_p (\hat{R} - \beta) x B_{\hat{R} - 1}^c (x)}{B_{\beta p}^c (x)} + \frac{k_p \beta x B_{\hat{R} - 1}^c (x)}{B_{\beta p}^c (x)} = j_p \alpha_p. \]

(25)

3.5 Proof of Eq. (19)

For \( V'_p \subset V_p \) with \( |V'_p| = w_p \) \((p = 1, 2)\), the expression of \( Z(V'_1, V'_2) \) defined in Eq. (3) for \( G_{2}(n, j_1, j_2, k_1, k_2) \) is given in the following lemma. The proof of this lemma is given in Appendix C.

**Lemma 6:** For \( p = 1, 2, \) let \( V'_p \subset V_p \) with \( |V'_p| = w_p \). Then

(1) For \( C' \subset C \) and \( p = 1, 2 \), let \( F(C', V'_p) \) denote the number of ways to connect between \( V'_p \) and \( C \) such that \((V'_p, C')\) and \((V'_p, C)\) are the odd and even connections, respectively. Then

\[ Z(V'_1, V'_2) = \sum_{\ell = 0}^m \sum_{|C'| = \ell} \sum_{p=1}^2 F(C', V'_p). \]

(26)

(2) For \( p = 1, 2 \),

\[ F(C', V'_p) = (j_p w_p)! (j_p (n_p - w_p))! \cdot \text{coef} \left( B_{\beta p}^c (x) B_{\beta p}^o (x)^{m-\ell}, x_j \right). \]

(27)

(3) \( Z(V'_1, V'_2) \) is expressed as

\[ Z(V'_1, V'_2) = \sum_{\ell = \ell_m (w_1, w_2)} \sum_{\ell = \ell_m (w_1, w_2)} \binom{m}{\ell} \left[ \frac{2}{p=1} \left( \binom{j_p w_p)! (j_p (n_p - w_p))! \cdot \text{coef} \left( B_{\beta p}^c (x) B_{\beta p}^o (x)^{m-\ell}, x_j \right) \right) \right] \]

(28)

where \( \ell_m (w_1, w_2) \) and \( \ell_m (w_1, w_2) \) are defined in Eqs. (20) and (21), respectively. \( \square \)

**(Proof of Eq. (19))** By substituting Eq. (28) into Eq. (2), we have

\[ \sum_{G \in G_2(n, j_1, j_2, k_1, k_2)} A_{w_1, w_2} \]

\[ = \sum_{\ell = \ell_m (w_1, w_2)} \sum_{\ell = \ell_m (w_1, w_2)} \binom{m}{\ell} \left[ \frac{2}{p=1} \left( \binom{j_p w_p)! (j_p (n_p - w_p))! \cdot \text{coef} \left( B_{\beta p}^c (x) B_{\beta p}^o (x)^{m-\ell}, x_j \right) \right) \right] \]

(28)

where the second equality is obtained by noting that \( Z(V'_1, V'_2) \) does not depend on a choice of \( V'_p \subset V_p \) with \( |V'_p| = w_p \) for \( p = 1, 2 \), and further by dividing both sides of the above equation by \( |G_2(n, j_1, j_2, k_1, k_2)| = (j_1 n_1)!, (j_2 n_2)! \), we can obtain Eq. (19).

3.6 Proof of Eq. (22)

By a similar argument of the proof of Eq. (11) in Sect. 3.3, we have

\[ a_{\alpha_1, \alpha_2} = \lim_{n \to \infty} \frac{1}{n} \log_2 \mathcal{A}_{\alpha_1, \alpha_2, n} \]

\[ = \hat{\gamma}_1 (1 - j_1) H_2 (\frac{\alpha_1}{\hat{\gamma}_1}) + \hat{\gamma}_2 (1 - j_2) H_2 (\frac{\alpha_2}{\hat{\gamma}_2}) + \max \beta \left\{ \hat{R} H_2 (\frac{\beta}{\hat{R}}) + \sum_{p=1}^2 \log_{\beta} B_{\beta p}^0 (x_p) B_{\beta p}^c (x_p) \right\} \]

(29)
where the maximum is taken over $\beta \in \mathbb{Q}$ satisfying $\beta_{\max}(\alpha_1, \alpha_2) \leq \beta \leq \beta_{\min}(\alpha_1, \alpha_2)$ where $\beta_{\max}(\alpha_1, \alpha_2)$ and $\beta_{\min}(\alpha_1, \alpha_2)$ are defined in Eqs. (23) and (24) respectively. In Eq. (29), for $p = 1, 2$,
\[
\lim_{n \to \infty} \frac{1}{n} \log_2 \left( \frac{B^{\beta}_{p} \left( x_p^B \right) B^{\beta}_{p} \left( x_p^B \right) (R - \beta)^n}{x_p^{\beta}} \right) = \log_2 \frac{B^{\beta}_{p} \left( x_p^B \right) B^{\beta}_{p} \left( x_p^B \right) (R - \beta)}{x_p^{\beta}}
\]
where the right hand side of the above equation is obtained by applying Lemma 3. Further $x_p$ is a unique positive solution of Eq. (25) and Eq. (25) is obtained by calculating $\frac{d^2}{dx^2} p(x) = \delta$ in Lemma 3 where $p(x) := B^{\beta}_{p} \left( x \right) B^{\beta}_{p} \left( x \right) (R - \beta)$ and $\delta := j_p x_p$.

By substituting the above result into Eq. (29), we obtain Eq. (22).

3.7 Stopping Set Distribution of $\mathcal{G}_1(n, j_1, k_1)$

Theorem 3: For $\mathcal{G}_1(n, j_1, k_1)$,

\[
\overline{A}_{a_1, a_2} = \frac{(n)}{(j_1\nu_1)} \sum_{m=0}^{\min\{m, \nu_2, n-m\}} T(m, w_2, \ell) \cdot \text{cof} \left(W_{k_1, m-w_2, f, 2\ell, w_2-w_2}(x), x^{j_1w_1}\right)
\]

where a function $T$ is defined in Eq. (10), and for $a, b, c, d \in \mathbb{Q}$,

\[
W_{a,b,c,d}(x) := \{(1+x)^a - ax^b \cdot (1+x)^c - 1 \cdot (1+x)^d \}\.
\]

If $\alpha_1 = 0$ and $0 < \alpha_2 < R$, then $a_{\alpha_1, \alpha_2} = -\infty$ and, otherwise,

\[
a_{\alpha_1, \alpha_2} = \gamma_1(1 - j_1)H_2 \left( \frac{a_1}{\gamma_1} \right) + \max_{\beta} \left\{ \alpha_2 H_2 \left( \frac{\beta}{\alpha_2} \right) + (R - \alpha_2) H_2 \left( \frac{\beta}{R - \alpha_2} \right) \right\} + \log_2 \left\{ \frac{W_{k_1, R-\alpha_2, 2\beta, \alpha_2, -\beta}(x_0)}{x_0^{j_1 a_1}} \right\}
\]

where $\gamma_1 := \frac{k_1}{j_1 n_1}$, $R := \frac{j_1 n_1}{j_1 n_1}$ and the maximum is taken over $\beta \in \mathbb{Q}$ satisfying $0 \leq \beta \leq \min\{\frac{a_1}{\alpha_2}, \alpha_2, R - \alpha_2\}$. Moreover, $x_0$ in Eq. (32) is a unique positive solution of

\[
(R - \alpha_2 - \beta)k_1 x \left(1 + x\right)^{k_1-1} - 1 = 0.
\]

3.8 Proof of Eq. (30)

For $V'_1 \subset V_1$ with $|V'_1| = w_1$, let

\[
Y_{sdd}(V'_1) := \sum_{V'_2 \subset V'_1 \atop |V'_2|=2} Z(V'_1, V'_2)
\]

where $Z(V'_1, V'_2)$ is defined in Eq. (4) for $\mathcal{G}_1(n, j_1, k_1)$. By a similar argument of Lemma 5 in Sect. 3.2, the expressions of $Y_{sdd}(V'_1)$ and $Z(V'_1, V'_2)$ are given in the following lemma. The proof of this lemma is given in Appendix D.

Lemma 7: Let $\ell$ be a non-negative integer and let $V'_1 \subset V_1$ with $|V'_1| = w_1$ and $V'_2 \subset X_{w_2, \ell}$ where $X_{w_2, \ell}$ is defined in Sect. 3.2. Further let $r_0, r_1$ and $r_2$ denote numbers of check nodes which are connected with $V'_2$ by no, one and more than two edges, respectively. Then

(1) $r_0 = m - w_2 - \ell$, $r_1 = 2\ell$ and $r_2 = w_2 - \ell$.

(2) $Z(V'_1, V'_2)$ is expressed as

\[
Z(V'_1, V'_2) = \left(j_1 w_1 \right) ! (j_1(n_1 - w_1)) ! \cdot \text{cof} \left(W_{k_1, m-w_2-2\ell, w_2-\ell}(x), x^{j_1w_1}\right).
\]

(3) $Y_{sdd}(V'_1)$ is expressed as

\[
Y_{sdd}(V'_1) = \left(j_1 w_1 \right) ! (j_1(n_1 - w_1)) ! \cdot \sum_{\ell=0}^{\min\{m, \nu_2, n-m\}} T(m, w_2, \ell) \cdot \text{cof} \left(W_{k_1, m-w_2-2\ell, w_2-\ell}(x), x^{j_1w_1}\right).
\]

3.9 Proof of Eq. (32)

By a similar argument of the proof of Eq. (11) in Sect. 3.3, we have

\[
\alpha_{a_1, a_2} = \lim_{n \to \infty} \frac{1}{n} \log_2 \overline{A}_{a_1, a_2,n} = \gamma_1(1 - j_1)H_2 \left( \frac{a_1}{\gamma_1} \right) + \max_{\beta} \left\{ \alpha_2 H_2 \left( \frac{\beta}{\alpha_2} \right) + (R - \alpha_2) H_2 \left( \frac{\beta}{R - \alpha_2} \right) \right\} + \log_2 \left\{ \frac{W_{k_1, R-\alpha_2, 2\beta, \alpha_2, -\beta}(x_0)}{x_0^{j_1 a_1}} \right\}
\]

where the maximum is taken over $\beta \in \mathbb{Q}$ satisfying $0 \leq \beta \leq \min\{\frac{a_1}{\alpha_2}, \alpha_2, R - \alpha_2\}$ and by applying Lemma 3,

\[
\lim_{n \to \infty} \frac{1}{n} \log_2 \left\{ \frac{W_{k_1, R-\alpha_2, 2\beta, \alpha_2, -\beta}(x_0)}{x_0^{j_1 a_1}} \right\} = \gamma_1(1 - j_1)H_2 \left( \frac{a_1}{\gamma_1} \right).
\]
where \( x_0 \) is a unique positive solution of Eq. (33) and Eq. (33) is obtained by calculating \( \frac{dp(y)}{dy} = \delta \) in Lemma 3 where \( p(x) = \: W_{1,R-a_0,\beta,2,a_0,\alpha-p}(x) \) and \( \delta = j_1a_1 \).

By substituting the above results into Eq. (37), we obtain Eq. (32).

### 3.10 Stopping Set Distribution of \( G_2(n, j_1, k_1, j_2, k_2) \)

**Theorem 4:** For \( G_2(n, j_1, k_1, j_2, k_2) \),

\[
\tilde{A}_{a_1,a_2} = \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \right) \sum \left( \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \right) \cdot k_2^j \\
\cdot \text{cof} \left( W_{k_1, k_2, \beta, \alpha}(x) \right) \\
\cdot \text{cof} \left( \left( (1 + x)^{k_2} - k_2 x - 1 \right)^j, x^{j_2w_2 - r} \right)
\]

where a function W is defined in Eq. (31) and the summation in the above equation is taken over \( r_0, r_1, r_2 \in \mathbb{Z} \) satisfying

\[
\begin{aligned}
\hat{r}_0 + \hat{r}_1 + \hat{r}_2 &= m, \\
\hat{r}_0 &\geq \max \left\{ 0, m - \frac{j_1 w_1 + j_2 w_2}{k_1}, m - j_2 w_2 \right\}, \\
\hat{r}_0 &\leq m - \frac{j_1 w_1}{k_1}, \\
\hat{r}_1 &\geq \max \left\{ 0, 2(m - \hat{r}_0) - j_2 w_2 \right\}, \\
\hat{r}_1 &\leq \min \left\{ j_1 w_1, j_2 w_2, \frac{k_2(m - \hat{r}_0) - j_2 w_2}{k_2 - 1} \right\}
\end{aligned}
\]

Moreover,

\[
\begin{aligned}
a_{a_1,a_2} &= \frac{j_1 k_1}{j_1 k_2 + j_2 k_1}, \\
\hat{\gamma}_2(x) &= \frac{j_1 k_1}{j_1 k_2 + j_2 k_1}, \\
\hat{\beta}_1, \hat{\beta}_2 \in \mathbb{Q} \text{ satisfying }
\end{aligned}
\]

\[
\begin{aligned}
\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 &= \hat{\beta}, \\
\hat{\beta}_0 &\geq \max \left\{ 0, \hat{\beta} - \frac{j_1 w_1 + j_2 w_2}{k_1}, \hat{\beta} - j_2 a_2 \right\}, \\
\hat{\beta}_0 &\leq \hat{\beta} - \frac{j_1 w_1}{k_2}, \\
\hat{\beta}_1 &\geq \max \left\{ 0, 2(\hat{\beta} - \hat{\beta}_0) - j_2 a_2 \right\}, \\
\hat{\beta}_1 &\leq \min \left\{ j_1 a_1, j_2 a_2, \frac{(k_2 - \hat{\beta}_0) - j_2 a_2}{k_2 - 1} \right\}
\end{aligned}
\]

Further \( x_1 \) and \( x_2 \) in Eq. (40) are unique positive solutions of Eqs. (42) and (43), respectively.
where the summation is taken over \( r_0, r_1, r_2 \in Z \) satisfying Eq. (39).

(Proof of Eq. (38)) By substituting Eq. (48) into Eq. (2), noting that \( Z(V', V') \) does not depend on ways to choose \( V_p \subset V_p \) with \( |V_p| = w_p \) for \( p = 1, 2 \), and dividing both side of Eq. (2) by \( G_2(n, j_1, k_1, j_2, k_2) = (j_1n_1)! (j_2n_2)! \), we obtain Eq. (38).

3.12 Proof of Eq. (40)

By a similar argument of the proof of Eq. (11) in Sect. 3.3, we have

\[
a_{\alpha_1, \alpha_2} = \lim_{n \to \infty} \frac{1}{n} \log_2 \mathcal{A}_{\alpha_1, n, \alpha_2} = \hat{\gamma}_1 (1 - j_1)H_2 \left( \frac{\alpha_1}{\gamma_1} \right) + \hat{\gamma}_2 (1 - j_2)H_2 \left( \frac{\alpha_2}{\gamma_2} \right)
\]

\[
+ \max_{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2} \left\{ RH_1 \left( \frac{\hat{\beta}_0}{R} \right) + \hat{\beta}_1 \log_2 k_2 \right\}
\]

\[
+ \lim_{n \to \infty} \frac{1}{n} \log_2 \text{coef} \left\{ W_{k, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2}(x), x^{\hat{\beta}_0 n} \right\}
\]

\[
+ \lim_{n \to \infty} \frac{1}{n} \log_2 \text{coef} \left\{ \left( |1 + x|^{k_2} - k_2 x - 1 \right)^{\hat{\beta}_1 n}, x^{(j_2 \alpha_2 - \hat{\beta}_1) n} \right\}
\]

(49)

where the maximum is taken over \( \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2 \in Q \) satisfying Eq. (41) and by applying Lemma 3,

\[
\lim_{n \to \infty} \frac{1}{n} \log_2 \text{coef} \left\{ W_{k, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2}(x), x^{\hat{\beta}_0 n} \right\}
\]

\[
= \log_2 \left\{ W_{k, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2}(x_1) \right\}
\]

\[
\lim_{n \to \infty} \frac{1}{n} \log_2 \text{coef} \left\{ \left( |1 + x|^{k_2} - k_2 x - 1 \right)^{\hat{\beta}_1 n}, x^{(j_2 \alpha_2 - \hat{\beta}_1) n} \right\}
\]

\[
= \log_2 \left\{ \left( |1 + x|^{k_2} - k_2 x - 1 \right)^{\hat{\beta}_1} \right\}
\]

\[
\frac{x_1^{\hat{\beta}_0 n}}{x_2^{j_2 \alpha_2 - \hat{\beta}_1}}
\]

where \( x_1 \) (resp. \( x_2 \)) is a unique positive solution of Eq. (42) (resp. Eq. (43)) and Eq. (42) (resp. Eq. (43)) is obtained by calculating \( \frac{p(x)}{p(x_1)} = \delta \) in Lemma 3 where \( p(x) := W_{k, \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2}(x) \) (resp. \( p(x) := |1 + x|^{k_2} - k_2 x - 1 |^{\hat{\beta}_1} \) and \( \delta := j_1 \alpha_1 \) (resp. \( \delta := j_2 \alpha_2 - \hat{\beta}_1 \)).

By substituting the above results, into Eq. (49), we obtain Eq. (40).

3.13 Numerical Examples

In this subsection, we show numerical examples of \( a_{\alpha_1, \alpha_2} \). We draw in Fig. 4 a graph of \( a_{\alpha_1, \alpha_2} \) for average stopping set distribution of \( G_2(n, 8, 10, 2, 2) \) for \( \alpha_1 \in [0, \gamma_1] \) and \( \alpha_2 \in [0, \gamma_2] \) where \( \gamma_1 = \frac{2.10}{8^2 + 2.10} = \frac{5}{9} \) and \( \gamma_2 = \frac{8}{8^2 + 2.10} = \frac{4}{9} \). Moreover, we magnify in Fig. 5 a part around the origin of Fig. 4. We see from Fig. 5 that \( a_{\alpha_1, \alpha_2} \) takes negative values for small \( \alpha_1 \) and \( \alpha_2 \). This implies that for such \( (\alpha_1, \alpha_2) \), the average number of stopping sets of \( G_2(n, 8, 10, 2, 2) \) decreases exponentially with the code length \( n \).

We define \( \alpha^* := \inf \{ \alpha > 0 : a_\alpha \geq 0 \} \). \( \alpha^* \) is called the typical minimum distance rate [4] (resp. the critical exponent stopping ratio [8]) if \( a_\alpha \) represents the asymptotic exponents for weight distribution (resp. stopping set distribution). We also denote by \( G_{\text{reg}}(n, j, k) \) an ensemble of regular LDPC codes based on an ensemble of bipartite graphs, each element of which has variable nodes of degree \( j \) and check nodes of degree \( k \). Then it is easily verified that \( G_2(n, j_1, k_1, j_2, k_2) \subset G_{\text{reg}}(n, j_1, k_1 + k_2) \). Moreover, by calculating \( a_{\alpha_0} \) for the weight distribution (resp. the stopping set distribution) of \( G_{\text{reg}}(n, j, k) \) given by the formula presented by Burshtein et al. [1] (resp. Orlitsky et al. [8]), we can obtain \( \alpha^* \) of \( G_{\text{reg}}(n, j, k) \). On the other hand, for two code ensembles we consider in this paper, by calculating \( a_{\alpha_1, \alpha_2} \) given in Eqs. (22) or (40) and using Eq. (6), we can obtain \( \alpha^* \) of those two code ensembles.

When we compare the typical minimum distance rates and the critical exponent stopping ratios of \( G_{\text{reg}}(n, 3, 6) \) and its two subsets \( G_2(n, 3, 4, 3, 2) \) and \( G_2(n, 3, 3, 3, 3) \), we can show that the typical minimum distance rates and the critical exponent stopping ratios of these three ensembles are the same values 0.02273 and 0.01799, respectively. This implies that for a given \( G_{\text{reg}}(n, j, k) \), there exist two-edge type LDPC code ensembles which are subsets of \( G_{\text{reg}}(n, j, k) \) and...
have the same performance in the sense of the typical minimum distance rate and the critical exponent stopping ratio.

Moreover, we compare in Fig. 6 the block error probability of an one-edge LDPC code belonging to $\mathcal{G}_{\text{reg}}(1000, 2, 4)$ and that of a two-edge type LDPC code belonging to $\mathcal{G}_2(1000, 2, 2, 2, 2)$. From Fig. 6, we can see that the block error probability of the code chosen from $\mathcal{G}_2(1000, 2, 2, 2, 2)$ is smaller than or equal to that of the code chosen from $\mathcal{G}_{\text{reg}}(1000, 2, 4)$.

4. Some Characteristics

In this section, we investigate some characteristics such as the symmetry and the conditions for zero of weight distributions of both code ensembles. These characteristics can be utilized for reducing computational complexity to calculate average weight distributions for two code ensembles. Moreover, we clarify the relation between the asymptotic exponents of two code ensembles.

4.1 Symmetry

Lemma 9: Let $\overline{A}_{w_1, w_2}$ and $\overline{A}_{w_1, w_2}$ denote the average weight distributions of $\mathcal{G}_1(n, j_1, k_1)$ and $\mathcal{G}_2(n, j_1, k_1, j_2, k_2)$, respectively.

1. We have $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ for $w_2 = 0, 1, \cdots, n_2$.

Moreover, if $k_1$ is even, then we have $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ for $w_1 = 0, 1, \cdots, n_1$.

2. If $k_1$ (resp. $k_2$) is even, then we have $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ for $w_1 = 0, 1, \cdots, n_1$ (resp. $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ for $w_2 = 0, 1, \cdots, n_2$).

Proof: (1) First, we show that $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$. By comparing the expression of $\overline{A}_{w_1, w_2}$ with that of $\overline{A}_{w_1, w_2}$, all we have to do is to prove $\ell_m(w_1, n_2 w_2) = \ell_m(w_1, w_2)$, $\ell_M(w_1, n_2 - w_2) = \ell_M(w_1, w_2)$, $\ell_m(w_1, n_2 - w_2) = \ell_m(w_1, w_2)$ and $\ell_M(w_1, n_2 - w_2) = \ell_M(w_1, w_2)$. Further by a straightforward calculation, we have $T(m, n_2 - w_2, \ell) = T(m, w_2, \ell)$.

Next, we show that if $k_1$ is even, then $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$. By comparing the expression of $\overline{A}_{w_1, w_2}$ with that of $\overline{A}_{w_1, w_2}$ and facts that $(\overline{w}_1, \overline{w}_2) = (\overline{w}_1, \overline{w}_2)$, we have to do is to prove $\ell_m(n_1 - w_1, 2) = \ell_m(w_1, 2)$, $\ell_M(n_1 - w_1, w_2) = \ell_M(w_1, w_2)$ and $\ell_m(n_1 - w_1, w_2) = \ell_m(w_1, w_2)$ and $\ell_M(n_1 - w_1, w_2) = \ell_M(w_1, w_2)$. By noting that $k_1$ is even and using Eqs. (8) and (9), it is easy to verify that $\ell_m(n_1 - w_1, 2) = \ell_m(w_1, 2)$ and $\ell_M(n_1 - w_1, w_2) = \ell_M(w_1, w_2)$. Equation (50) is also shown as follows. For a polynomial $p(x)$, it is easy to verify that $\overline{w}_2(x^N)$ where $N := \maxdeg(p(x))$. Then it is sufficient to show that $B_{w_1}^1(x^{2t})B_{w_1}^1(x^{m-2t}, x^{n_1-n_2})$ is symmetric and this is shown as follows. It is easy to verify that the product of two polynomials which are symmetric is symmetric. Further both $B_{w_1}^1(x)$ and $B_{w_1}^1(x)$ are symmetric. Therefore $B_{w_1}^1(x^{2t})B_{w_1}^1(x^{m-2t}, x^{n_1-n_2})$ is symmetric.

2. By a similar argument of the proof of (1) of this lemma, we can prove (2) of this lemma.

From Lemma 9, we see that the weight distributions of both $\mathcal{G}_1(n, j_1, k_1)$ and $\mathcal{G}_2(n, j_1, k_1, j_2, k_2)$ have some symmetric properties (i.e. $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ or $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$) with respect to $w_1$ or $w_2$ under some conditions.

The symmetric properties help to calculate $\overline{A}_{w_1, w_2}$ given in Eqs. (7) and (19) for the finite code length. For $(w_1, w_2)$ satisfying the conditions in Lemma 9, $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ or $\overline{A}_{w_1, w_2} = \overline{A}_{w_1, w_2}$ holds. Therefore we can obtain $\overline{A}_{w_1, w_2}$ for $0 \leq w_1 \leq n_1$ (or $0 \leq w_2 \leq n_2$) by calculating those for $0 \leq w_1 \leq \frac{n_1}{2}$ (or $0 \leq w_2 \leq \frac{n_2}{2}$).

4.2 Conditions for Zero of Weight Distribution

In this subsection, we investigate the conditions on $w_1$ and $w_2$ such that $\overline{A}_{w_1, w_2} = 0$ where $\overline{A}_{w_1, w_2}$ represents the weight distribution of $\mathcal{G}(n, j_1, k_1)$.

Lemma 10: Let $k_1$ be odd. For $\mathcal{G}_1(n, j_1, k_1)$, the conditions on $w_1$ and $w_2$ such that $\overline{A}_{w_1, w_2} = 0$ where $\overline{A}_{w_1, w_2}$ represents the weight distribution of $\mathcal{G}(n, j_1, k_1)$ are given as

\[
\begin{align*}
0 \leq w_1 \leq n_1, & \quad 0 \leq w_2 \leq n_2, \\
0 \leq w_1 \leq n_1, & \quad 0 \leq w_2 \leq n_2.
\end{align*}
\]

Proof: If $k_1$ is odd, $\ell_m(w_1, w_2)$ and $\ell_M(w_1, w_2)$ is given as
$$\ell_m(w_1, w_2) = \max \left\{ \frac{j_1 w_1}{2}, w_2, m - w_2 \right\}.$$  
$$\ell_M(w_1, w_2) = \min \left\{ \frac{j_1 w_1}{2}, w_2, m - w_2 \right\}.$$  

Moreover, if $\ell_m(w_1, w_2) > \ell_M(w_1, w_2)$, then $\mathcal{A}_{w_1, w_2} = 0$. Therefore for $w_1$ and $w_2$ satisfying one of the following conditions, $\mathcal{A}_{w_1, w_2} = 0$.

$$\frac{j_1}{2} \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right) > \frac{j_1 w_1}{2}, \quad \text{(53)}$$  
$$\frac{j_1}{2} \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right) > w_2, \quad \text{(54)}$$  
$$\frac{j_1}{2} \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right) > m - w_2. \quad \text{(55)}$$  

Note that there does not exist $w_i$ satisfying Eq. (53). By Eqs. (54) and (55), we can obtain Eqs. (51) and (52).

By a similar argument of Lemma 10, for $\mathcal{G}_2(n, j_1, k_1, j_2, k_2)$, we have the following conditions on $w_1$ and $w_2$ such that $\mathcal{A}_{w_1, w_2} = 0$.

**Lemma 11:** For $\mathcal{G}_2(n, j_1, k_1, j_2, k_2)$, the conditions on $w_1$ and $w_2$ such that $\mathcal{A}_{w_1, w_2} = 0$, where $\mathcal{A}_{w_1, w_2}$ represents the weight distribution of $\mathcal{G}_2(n, j_1, k_1, j_2, k_2)$, are given as

1. If $k_1$ is even and $k_2$ is odd,  
   $$\frac{j_1 w_1}{2} < \frac{j_2 (w_2 - \frac{k_1 - 1}{k_1} n_2)}{2}, \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2,$$
   or  
   $$j_1 w_1 > j_1 n_1 - j_2 \left( w_2 - \frac{k_1 - 1}{k_1} n_2 \right), \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2.$$  

2. If $k_1$ is odd and $k_2$ is even,  
   $$\frac{j_1 w_1}{2} < j_1 \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right), \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2,$$
   or  
   $$j_1 w_1 > j_1 n_2 - j_1 \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right), \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2.$$  

3. If $k_1$ is odd and $k_2$ is odd,  
   $$\frac{j_1 w_1}{2} < j_1 \left( w_1 - \frac{k_1 - 1}{k_1} n_1 \right), \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2,$$
   or  
   $$j_1 w_1 > j_1 \left( w_2 - \frac{k_1 - 1}{k_1} n_2 \right), \quad 0 \leq w_1 \leq n_1, 0 \leq w_2 \leq n_2.$$  

Lemma 10 and Lemma 11 show that there is no need to calculate Eqs. (7) and (19) for $(w_1, w_2)$ satisfying the conditions for zero of the weight distribution because it is guaranteed that $\mathcal{A}_{w_1, w_2} = 0$ for those $(w_1, w_2)$.

### 4.3 Relation between the Asymptotic Exponents of Two Code Ensembles

Let $j$ and $k$ be two positive integers. In this subsection, we investigate the relation between $\mathcal{G}_2(n, j, k)$ and $\mathcal{G}_2(n, j, k, 2, 2)$ with respect to the asymptotic exponents for both the weight and stopping set distributions.

**Theorem 5:** Let $a_{\alpha_1, \alpha_2}$ and $a'_{\alpha_1, \alpha_2}$ denote the asymptotic exponents of $\mathcal{G}_1(n, j, k)$ and $\mathcal{G}_2(n, j, k, 2, 2)$ for the weight or stopping set distributions, respectively. Then $a_{\alpha_1, \alpha_2} = a'_{\alpha_1, \alpha_2}$.

**Proof:** We only prove this theorem for weight distribution because by a similar argument of the proof of that for weight distribution, we can prove that for stopping set distribution.

It is easy to verify that $\gamma_1 = \gamma_1$ and $\gamma_2 = R$ where $\gamma_1, \gamma_2$ and $\gamma_1, \gamma_2, R$ are defined in Theorem 1 and 2, respectively. By substituting $j_2 = k_2 = 2$ into Eq. (22), $a'_{\alpha_1, \alpha_2}$ is written as

$$a'_{\alpha_1, \alpha_2} = \gamma_1 (1 - j) H_2 \left( \frac{\alpha_1}{\gamma_1} \right) - RH_2 \left( \frac{\alpha_2}{R} \right)$$  
$$+ \max_{\beta} \left\{ RH_2 \left( \frac{\beta}{R} \right) + \log_2 \frac{B^2_1(x_1, x_2) B^2_2(x_1, x_2) \beta^{R - \beta}}{x_1^2} \right\}$$  
$$+ \log_2 \frac{B^2_1(x_1, \beta^{R - \beta})}{x_1^2} \right\} \quad \text{(56)}$$

where the maximum is taken over $\beta \in \mathbb{Q}$ satisfying $\beta_\alpha(a_1, a_2) \leq \beta \leq \beta_{\mathcal{M}}(a_1, a_2)$ where $\beta_\alpha(a_1, a_2)$ and $\beta_{\mathcal{M}}(a_1, a_2)$ are defined in Eqs. (23) and (24) respectively, and by noting $j_2 = k_2 = 2$, given as

$$\beta_{\alpha} (a_1, a_2) \quad \text{(57)}$$  
$$\beta_{\mathcal{M}} (a_1, a_2) \quad \text{(58)}$$

Let $\beta' := \frac{\beta}{R}$, $\beta'_{\alpha} (a_1, a_2) := \frac{\beta_\alpha(a_1, a_2)}{R}$ and $\beta'_{\mathcal{M}} (a_1, a_2) := \frac{\beta_{\mathcal{M}}(a_1, a_2)}{R}$. First we show the following lemma. The proof of this lemma is given in Appendix F.

**Lemma 12:** In the right hand side of Eq. (56),

1. $\beta'_{\alpha} (a_1, a_2) = \beta_{\alpha} (a_1, a_2)$ and $\beta'_{\mathcal{M}} (a_1, a_2) = \beta_{\mathcal{M}} (a_1, a_2)$ where $\beta_\alpha(a_1, a_2)$ and $\beta_{\mathcal{M}}(a_1, a_2)$ are defined in Eqs. (12) and (13).

2. $x_2 = \sqrt{\frac{\alpha_1 - \alpha_2}{R - \alpha_2}}$ where $x_2$ is a unique positive solution of Eq. (25) for $p = 2$.

3. 

$$-RH_2 \left( \frac{\alpha_2}{R} \right) + RH_2 \left( \frac{\beta'}{R} \right) + \log_2 \frac{B^2_1(x_2, \beta') B^2_2(x_1, \beta') \beta'^{R - \beta}}{x_1^2}$$  
$$= \alpha_2 H_2 \left( \frac{\beta'}{R} \right) + (R - \alpha_2) H_2 \left( \frac{\beta'}{R - \alpha_2} \right). \quad \text{(59)}$$
Appendix A: Proof of Lemma 4

(Proof of Theorem 5) In Eq. (56), \( x_1 \) is a unique positive solution of Eq. (25) for \( p = 1 \) and Eq. (25) coincides with Eq. (14). Therefore \( x_1 \) coincides with \( x_0 \) which is a unique positive solution of Eq. (14). Further by substituting \( \beta = 2 \beta^2 \) and results of (1) and (3) of Lemma 12 into Eq. (56), we have \( a_{1,2}^1 = a_{1,2}^2 \).

5. Conclusion

In this paper, we have explicitly formulated the average weight and the stopping set distributions \( (A_{w_{1,2}}) \) and their asymptotic exponents \( (a_{1,2}) \) of \( \mathcal{G}_1(n, j_1, k_1) \) and \( \mathcal{G}_2(n, j_1, k_1, j_2, k_2) \). We have also shown the symmetry and the conditions for zero of the weight distributions of two code ensembles. Further we have investigated the relation between \( \mathcal{G}_1(n, j_1, k_1) \) and \( \mathcal{G}_2(n, j_1, k_1, j_2, k_2) \) from the perspectives of the weight and the stopping set distributions.

Though, in this paper, we have investigated the average weight and the stopping set distributions and their asymptotic exponents of ensembles of bipartite graphs, each of which has two edge types, one of future works is to extend the arguments in this paper to ensembles of multi-edges type LDPC code ensembles [9].

References


Appendix A: Proof of Lemma 4

(1) First, we show \( \mathbb{E}[w_{2,\ell}] = T(m, w_2, \ell) \) for \( w_2 = 0 \) and \( w_2 = m \). By noting \( X_0 = \emptyset \) and \( X_m = \mathbb{E}[2] = X_{m,1} \), we have \( \mathbb{E}[w_{2,\ell}] = 1 = T(m, w_2, \ell) \).

Next, we consider \( w_2 \) such that \( 1 \leq w_2 < m \). Let \( U' := \left\{ v_{2.1}, v_{2.2}, \ldots, v_{2.a_1-1} \right\}, \ (a_1 > 1), \end{array} \) 

and for \( i = 1, 2, \ldots, \ell - 1, \)

\( S' := \left\{ v_{2.b_i+1}, v_{2.b_i+2}, \ldots, v_{2.a_1-1} \right\} \)

and \( S'_i := V_2((\bigoplus_{i=0}^{n-1} S_i) \bigoplus (\bigoplus_{i=1}^{n-1} S'_i) \bigoplus U \bigoplus U') \).

Moreover, let \( u := |U| \) and \( u' := |U'| \), and for \( i = 1, 2, \ldots, \ell, \ s_i := |S_i| \) and \( s'_i := |S'_i| \). Note that \( U, U' \) and \( S'_i \) can be empty sets. Therefore \( u, u' \) and \( s_i \) satisfy

\[ s_1 + s_2 + \cdots + s_\ell + u = w_2, \]

\[ s'_1 + s'_2 + \cdots + s'_\ell = m - w_2, \]

\( s_1, s_2, \ldots, s_\ell \geq 1, u \geq 0, \]

\( s'_1, s'_2, \ldots, s'_\ell \geq 1 \)

and the number of \( V_2' \in X_{w_2,\ell} \) satisfying above conditions is given as

\[ \binom{w_2}{\ell} (m - w_2 - 1) (\ell - 1) \tag{A-1} \]

Case 2: If \( v_{2,1} \notin S_1 \), then \( U' = \emptyset \) and \( S'_i \neq \emptyset \), that is, \( u' = 0 \) and \( s'_i \geq 1 \), respectively. Therefore \( u, s_i \) and \( s'_i \) satisfy

\[ s_1 + s_2 + \cdots + s_\ell = w_2, \]

\[ s'_1 + s'_2 + \cdots + s'_\ell + u' = m - w_2, \]

\( s_1, s_2, \ldots, s_\ell \geq 1, \]

\( s'_1, s'_2, \ldots, s'_\ell \geq 1 \)

and the number of \( V_2' \in X_{w_2,\ell} \) satisfying above conditions is given as

\[ \binom{w_2 - 1}{\ell} (m - w_2) (\ell - 1) \tag{A-2} \]

By Eqs. (A-1) and (A-2), \( \mathbb{E}[w_{2,\ell}] \) is given as

\[ \mathbb{E}[w_{2,\ell}] = \binom{w_2}{\ell} (m - w_2 - 1) + \binom{w_2 - 1}{\ell} (m - w_2) \]

\[ = \frac{\ell m}{w_2(m - w_2)} \left( \frac{m - w_2}{\ell - 1} \right) \]

\[ = T(m, w_2, \ell). \]

(2) We show \( X_{w_2} = \bigoplus_{w_2=m}^{\min(m, m-w_2)} X_{w_2,\ell} \). Físt, for \( w_2 = 0, m \), clearly \( X_0 = X_{m,0} \) and \( X_m = X_{m,1} \) hold. Next, for \( 1 \leq w_2 < m \), clearly, for any \( V_2' \in X_{w_2} \), there exists a positive integer \( \ell \) such that \( V_2' \) is divided into \( \ell \) sets and for two different positive integers \( \ell, \ell' \), \( X_{w_2,\ell} \cap X_{w_2,\ell'} = \emptyset \). Thus the remaining part of the proof is that the maximum value which \( \ell \) can take is equal to \( \min(w_2, m-w_2) \).
Recall that $\ell$ represents the number of sets $S_i$ of $V'_2$. For $V'_2 \in X_{w_2, \ell}$, when each set $S_i$ consists of only one element, $\ell$ takes $w_2$ and $\ell$ can also regard as the number of sets $S'_i$ of $V_2 | V'_2$. By noting that $|V_2 | V'_2| = m - w_2$, when each set $S'_i$ consists of only one element, $\ell$ takes $m - w_2$. Therefore the maximum of $\ell$ is equal to $\min\{w_2, m - w_2\}$.

Appendix B: Proof of Lemma 5

Let $r_{\text{odd}}$ and $r_{\text{even}}$ denote the number of check nodes which are connected with each $c \in C$ by an odd and even number of edges, respectively.

(1) We show that $r_{\text{odd}} = 2\ell$ and $r_{\text{even}} = m - 2\ell$.

Figure A-1 shows the situation of connection between $S_i$, $(i = 1, 2, \ldots, \ell)$ and its neighbor check nodes. In Fig. A-1, a filled circle represents a variable node in $S_i$ and a filled square represents a check node which connects to $S_i$ by a single edge. As we see in Fig. A-1, for $i = 1, 2, \ldots, \ell$, there are two check nodes which are connected with $S_i$ (i.e. $V'_2$) by a single edge (i.e. an odd number of edges) and all remaining check nodes are connected with $V'_2$ by zero or two of edges (i.e. an even number of edges). Thus $r_{\text{odd}}$ and $r_{\text{even}}$ are given by

$$r_{\text{odd}} = \sum_{i=1}^{\ell} 2 = 2\ell,$$
$$r_{\text{even}} = m - r_{\text{odd}} = m - 2\ell,$$

respectively.

(2) For $V'_2 \in X_{w_2, \ell}$, let $C'$ denote the set of all check nodes, each of which is connected with $V'_2$ by an odd number of edges. Note that $(V'_2, C')$ and $(V'_2, C \setminus C')$ are the odd and even connections, respectively. Then for $V'_1 \in V_1$ with $|V'_1| = w_1$ and $V'_2 \in X_{w_2, \ell}$,

$$Z(V'_1, V'_2) = \cdots$$

By noting that there are $j_1 w_1$ variable sockets on $V'_1$, the number of ways to choose $j_1 w_1$ check sockets such that each check node $c \in C$ and $(V'_1, C')$ have odd and even numbers of check sockets in chosen $j_1 w_1$ check sockets respectively is given by

$$\text{coef} \left( B_{k_1}^0 (x)^{j_1 w_1} B_{k_1}^1 (x)^{r_{\text{even}} - j_1 w_1} \right).$$

and when $j_1 w_1$ variable sockets on $V_1$ are connected with chosen $j_1 w_1$ check sockets, $(V'_1, C')$ and $(V'_1, C \setminus C')$ become the odd and even connections. The number of such a connection is equal to $(j_1 w_1)!$ and the number of ways to connect between remaining $(j_1(n_1 - w_1))$ variable and check sockets is equal to $(j_1(n_1 - w_1))!$. Therefore

$$Z(V'_1, V'_2) = (j_1 w_1)! (j_1(n_1 - w_1))! \cdot \text{coef} \left( B_{k_1}^0 (x)^{2\ell} B_{k_1}^1 (x)^{m - 2\ell} \right).$$

(3) For $a, b, c, d \in \mathbb{Z}$, let

$$D_{a,b,c,d} := \text{coef} \left( B_{a}^0 (x)^c B_{b}^1 (x)^d \right)$$

and we consider the range of $\ell$ such that $D_{a,b,c,d} = 0$. Since $\ell$ is the smallest degree of $x$ in the expansion of $B_{a}^0 (x)^c B_{b}^1 (x)^d$, for $\ell > \frac{b - c}{2}$, we have $D_{a,b,c,d} = 0$. Also when $k_1$ is even, the largest degree of $x$ is

$$(k_1 - 1) \cdot 2\ell + k_1 \cdot (m - 2\ell) = j_1 n_1 - 2\ell$$

and for $j_1 n_1 - 2\ell < j_1 w_1$, that is, $\ell < \frac{j_1(n_1-w_1)}{k_1}$, we have $D_{a,b,c,d} = 0$. Similarly, when $k_1$ is odd, the largest degree of $x$ is

$$k_1 \cdot 2\ell + (k_1 - 1) \cdot (m - 2\ell) = 2\ell + (k_1 - 1)m$$

and for $\ell + (k_1 - 1)m < j_1 w_1$, that is, $\ell < \frac{j_1 w_1}{k_1 - 1}$, we have $D_{a,b,c,d} = 0$. Thus Combining the range of $\ell$ such that $D_{a,b,c,d} = 0$ with $0 \leq \ell \leq \min\{w_2, m - w_2\}$, the minimum and maximum values of $\ell$ are equal to $\ell_{\text{min}}(w_1, w_2)$ and $\ell_{\text{max}}(w_1, w_2)$ where $\ell_{\text{min}}(w_1, w_2)$ and $\ell_{\text{max}}(w_1, w_2)$ are defined in Eqs. (8) and (9), respectively.

Now, we derive the expression of $Y_{\text{odd}}(V'_1)$. By substituting results of Lemma 4 and Eq. (16) into Eq. (15) and noting the range of $\ell$ described above,
\[ Y_{wd}(V'_1) = \sum_{V'_2 \subseteq V_2; |V'_2| = 2} Z(V'_1, V'_2) \]
\[ = \sum_{\ell = \ell_0}^{\ell_{\max}} \sum_{V'_2 \subseteq X_{\ell-\ell}} Z(V'_1, V'_2) \]
\[ = \sum_{\ell = \ell_0}^{\ell_{\max}} \sum_{\ell' = \ell_0}^{\ell_{\max}} \prod_{(i, w_i) \in E_{\ell', \ell}} \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-2\ell}, x^{i w_i} \right) \]
\[ = (j_1 w_1) (j_1 (n_1 - w_1))! \]
\[ \sum_{\ell = \ell_0}^{\ell_{\max}} T(m, w_2, \ell) \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-2\ell}, x^{i w_i} \right) \]

where the second equality is obtained by (2) of Lemma 4, and the third equality is obtained by substituting Eq. (16) and noting that \( Z(V'_1, V'_2) \) does not depend on ways to choose \( V'_2 \in X_{\ell-\ell} \). Further the fourth equality is obtained by substituting (1) of Lemma 4.

Appendix C: Proof of Lemma 6

For \( p = 1, 2 \), let \( V'_p \subseteq V_p \) with \( |V'_p| = w_p \).

1. \( Z(V'_1, V'_2) \) is transformed into

\[ Z(V'_1, V'_2) = \{|G \in G_2(n, j_1, j_2, k_2) : (V'_p \cup V'_2, C) \text{ is the even connection.}| \] \[ = \sum_{C \subseteq C} \{\{G \in G_2(n, j_1, j_2, k_2) : (V'_p, C \setminus C') \text{ is odd and even connections, respectively.}\} \] \[ = \sum_{\ell = \ell_0}^{\ell_{\max}} T(m, w_2, \ell) \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-2\ell}, x^{i w_i} \right) \]

For \( p = 1, 2 \), the number of ways of connect between \( V_p \) and \( C \) such that \( (V'_p, C') \) and \( (V'_p, C \setminus C') \) are odd and even connections is equal to \( F(C', V'_p) \) where \( F(C', V'_p) \) is defined in (1) of Lemma 6. Thus

\[ \{|G \in G_2(n, j_1, j_2, k_2) : (V'_p, C') \text{ and } (V'_p, C \setminus C') \text{ is odd and even connections, respectively.}| \] \[ = \sum_{\ell = \ell_0}^{\ell_{\max}} T(m, w_2, \ell) \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-2\ell}, x^{i w_i} \right) \]

Further by noting \( \{C' \subseteq C\} = \bigoplus_{\ell=0}^{\ell_{\max}} \{C' \subseteq C : |C'| = \ell\} \) where \( \bigoplus \) represents the union of disjoint sets, we have

\[ Z(V'_1, V'_2) = \sum_{C \subseteq C} \prod_{p=1}^{2} F(C', V'_p) \]
\[ = \sum_{\ell = \ell_0}^{\ell_{\max}} \sum_{\ell' = \ell_0}^{\ell_{\max}} \prod_{p=1}^{2} F(C', V'_p) \]

(2) By a similar argument in the proof of (2) of Lemma 5 in Appendix B, for \( p = 1, 2 \), the expression of \( F(C', V'_p) \) is given as

\[ F(C', V'_p) = (j_p w_p)! (j_p (n_p - w_p))! \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-m-\ell}, x^{j_p w_p} \right) \]

(3) First, for \( p = 1, 2 \), we consider the range of \( \ell \) such that \( D_{k_p}(\ell, m - \ell, j_p w_p) = 0 \) where a function \( D \) is defined in Eq. (A-3) and \( D_{k_p}(\ell, m - \ell, j_p w_p) \) is given as

\[ D_{k_p}(\ell, m - \ell, j_p w_p) \]
\[ = \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-m-\ell}, x^{j_p w_p} \right) \]

Since the smallest and largest degrees of \( x \) in the expansion of \( B_{k_p}^p(x) B_{k_p}^p(x)^{-m-\ell} \) are equal to \( \ell \) and

\[ \{ k_p m - \ell, (k_p : \text{even}) \}
\[ \{ (k_p - 1) m + \ell, (k_p : \text{odd}) \}

respectively; if \( \ell > j_p w_p \), \( k_p m - \ell < j_p w_p \) and \( k_p \) is even, or if \( (k_p - 1) m + \ell < j_p w_p \) and \( k_p \) is odd, we have \( D_{k_p}(\ell, m - \ell, j_p w_p) = 0 \). Therefore the minimum and maximum values of \( \ell \) such that the terms in the summations in Eq. (26) is not equal to zero are equal to \( \ell_{\min}(w_1, w_2) \) and \( \ell_{\max}(w_1, w_2) \) where \( \ell_{\min}(w_1, w_2) \) and \( \ell_{\max}(w_1, w_2) \) are defined in Eqs. (20) and (21), respectively.

Now we derive the expression of \( Z(V'_1, V'_2) \). By substituting Eq. (27) into Eq. (26) and noting that \( F(C', V'_p) \) does not depend on ways to choose \( C' \subseteq C \) with \( |C'| = \ell \), we obtain

\[ Z(V'_1, V'_2) \]
\[ = \sum_{\ell = \ell_0}^{\ell_{\min}} \sum_{k_p m - \ell, (k_p : \text{even})}^{\ell_{\max}} \sum_{(k_p - 1) m + \ell, (k_p : \text{odd})}^{\ell_{\max}} \prod_{p=1}^{2} F(C', V'_p) \]
\[ = \sum_{\ell = \ell_0}^{\ell_{\max}} \sum_{\ell' = \ell_0}^{\ell_{\max}} \prod_{p=1}^{2} (j_p w_p)! (j_p (n_p - w_p))! \cdot \text{coef} \left( B_{k_p}^p(x) B_{k_p}^p(x)^{-m-\ell}, x^{j_p w_p} \right) \]

Appendix D: Proof of Lemma 7

(1) As we see in Fig. A-1, for all \( i = 1, 2, \cdots, \ell \), the number of check nodes which are connected with \( S_i (S_i \cup U) \) for \( i = 1 \) is equal to \( |S_i| + 1 (|S_i| = \ell) \) and there are \( 2 \) and \( |S_i| - 1 (|S_i| = \ell) \) check nodes which are connected with \( S_j \) by one and two edges, respectively. By the definitions of \( r_0, r_1 \) and \( r_2 \) and noting that \( |S| = \sum_{i=1}^{\ell} |S_i| = w_2 \), we have

\[ r_1 = 2 + \sum_{i=1}^{\ell} 2 = 2 \ell \]
\[ r_2 = (|S| + 1) - 1 + \sum_{i=1}^{\ell} 2 = w_2 - \ell \]
\[ r_0 = m - r_1 - r_2 = m - w_2 - \ell \.]
(2) Given \( V'_1 \in X_{c_2} \), let \( C_0, C_1 \) and \( C_2 \) denote the set of check nodes which are connected with \( V'_2 \) by one, one and two edges, respectively. Then in order to make \( V'_1 \cup V'_2 \) be a stopping set, \( V'_1 \) must be connected with \( c \in C_0, C_1 \) and \( C_2 \) by zero or more than two, more than one and any number of edges, respectively. Thus for \( V'_1 \subset V_1 \) with \( |V'_1| = w_1 \) and \( V'_2 \subset X_{c_2} \),

\[
Z(V'_1, V'_2) = \left( j_1(w_1)! \right) (j_1(n_1 - w_1)!) \\
\cdot \text{coef} \left[ \frac{1}{(1 + x)^{k_1} - k_1 x^p} \right]_{x^{i(w_1)}} \\
\cdot \text{coef} \left[ \frac{1}{(1 + x)^{k_11} - x^{i(w_1)}} \right]
\]

where the second equality is obtained by (1) of Lemma 7.

(3) First, we consider the range of \( \ell \) such that \( \text{coef} \left( W_{k_1m - \ell, 2, \ell} r(x), x^{i(w_1)} \right) = 0 \). Since the smallest and largest degrees of \( x \) in the expansion of \( W_{k_1m - \ell, 2, \ell} r(x) \) are \( 2 \ell \) and \( k_1(m - w_2 - \ell) + 2 \ell + (w_2 + 0) \), we have \( k_1m = j_1n_1(\geq j_1w_1) \) respectively, for \( \ell > \frac{i(w_1)}{2} \), we have \( \text{coef} \left( W_{k_1m - \ell,2,\ell} r(x), x^{i(w_1)} \right) = 0 \). Therefore combining the range of \( \ell \) such that \( \text{coef} \left( W_{k_1m - \ell,2,\ell} r(x), x^{i(w_1)} \right) \neq 0 \), that is, \( \ell \leq \frac{i(w_1)}{2} \) with the range of \( \ell \) such that \( T(m, w_2, \ell) \neq 0 \), that is, \( 0 \leq \ell \leq \min[w_2, m - w_2] \), we consider only \( \ell \) such that \( 0 \leq \ell \leq \min\left[ \frac{i(w_1)}{2}, w_2, m - w_2 \right] \). By substituting Eq. (35) into Eq. (34) and a similar argument of the proof of (3) of Lemma 5, we obtain Eq. (36).

**Appendix E: Proof of Lemma 8**

For \( p = 1, 2 \), let \( V'_p \subset V_p \) with \( |V'_p| = w_p \).

(1) For three subsets \( \hat{C}_0, \hat{C}_1 \) and \( \hat{C}_2 \) of \( C \) such that \( \hat{C}_0 \oplus \hat{C}_1 \oplus \hat{C}_2 = C \), assume that \( V'_2 \not\subset \hat{C}_0, V'_2 \not\subset \hat{C}_1 \) and \( V'_2 \not\subset \hat{C}_2 \). Note that the number of ways to connect between \( V_2 \) and \( C \) satisfying the above assumption is equal to \( F_2(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_2) \). Under above conditions, in order to make \( V'_1 \cup V'_2 \) be a stopping set, by the definition of stopping set in Definition 1, it must be \( V'_1 \not\subset \hat{C}_0, V'_1 \not\subset \hat{C}_1 \) and \( V'_1 \not\subset \hat{C}_2 \), and the number of ways to connect between \( V_1 \) and \( C \) satisfying the above condition is equal to \( F_1(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_1) \). Thus

\[
Z(V'_1, V'_2) = \left( \left( \sum_{\hat{C}_0, \hat{C}_1, \hat{C}_2 \subset C} |G \in G_2(n, j_1, k_1, j_2, k_2) : V'_1 \cup V'_2 \not\subset \hat{C}_1 \right) \right) \\
= \sum_{\hat{C}_0, \hat{C}_1, \hat{C}_2 \subset C} \left( |G \in G_2(n, j_1, k_1, j_2, k_2) : \right) \\
V'_1 \not\subset \hat{C}_0, V'_1 \not\subset \hat{C}_1, V'_1 \not\subset \hat{C}_2, \\
V'_2 \not\subset \hat{C}_0, V'_2 \not\subset \hat{C}_1, V'_2 \not\subset \hat{C}_2 \)
\]

(2) Since by the similar argument to derive the expression of \( F_1(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_1) \), we can derive that of \( F_2(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_2) \), only derive the expression of \( F_1(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_1) \).

There are \( j_1w_1 \) variable sockets on \( V'_1 \). Now we choose \( j_1w_1 \) check sockets such that \( V'_1 \not\subset \hat{C}_0, V'_1 \not\subset \hat{C}_1 \) and \( V'_1 \not\subset \hat{C}_2 \) when chosen \( j_1w_1 \) check sockets are connected with \( j_1w_1 \) variable sockets \( V'_1 \). By noting \( |\hat{C}_0| = r_0, |\hat{C}_1| = r_1 \) and \( |\hat{C}_2| = r_2 \), the number of ways to choose check sockets satisfying the above condition is given as

\[
\text{coef} \left( \left( (1 + x)^{k_1} - k_1 x^p \right) \left( (1 + x)^{k_11} - x^{i(w_1)} \right) \right)_{x^{i(w_1)}} = \text{coef} \left( W_{k_1, r_0, r_1, r_2} (x), x^{i(w_1)} \right)
\]

where a function \( W \) is defined in Eq. (31). Further the numbers of ways such that \( j_1w_1 \) and \( j_1(n_1 - w_1) \) variable sockets on \( V'_1 \) and \( V_1 \setminus V'_1 \) are connected with chosen \( j_1w_1 \) and remaining \( j_1(w_1 - n_1) \) check sockets are \((j_1w_1)! \) and \((j_1(n_1 - w_1))! \), respectively. Therefore we have

\[
F_1(\hat{C}_0, \hat{C}_1, \hat{C}_2, V'_1) = \left( j_1w_1 \right)! \left( j_1(w_1 - n_1) \right)!
\cdot \text{coef} \left( W_{k_1, r_0, r_1, r_2} (x), x^{i(w_1)} \right)
\]

(3) By investigating conditions on \( r_0, r_1 \) and \( r_2 \) such that \( \text{coef}(W_{k_1, r_0, r_1, r_2} (x), x^{i(w_1)}) = 0 \) and \( \text{coef}(1 + x)^{k_1} - k_2 x - 1) = 0 \), we can obtain conditions on \( r_0, r_1 \) and \( r_2 \) which are given in Eq. (39) and therefore by substituting Eqs. (46) and (47) into Eq. (44), we have

\[
Z(V'_1, V'_2)
\]

where the first summations in the last equation is taken over \( \hat{C}_0, \hat{C}_1 \) and \( \hat{C}_2 \) satisfying Eq. (45).
Appendix F: Proof of Lemma 12

(1) Let $\beta_m'(\alpha_1, \alpha_2) := \frac{\beta_m(\alpha_1, \alpha_2)}{2}$ and $\beta_m''(\alpha_1, \alpha_2) := \frac{\beta_m(\alpha_1, \alpha_2)}{2}$. Then by Eqs. (57) and (58), we can get

\[
\beta_m'(\alpha_1, \alpha_2) = \begin{cases} 
0, & (k_1 : \text{even}), \\
\max[0, \frac{\beta}{2} (\alpha_1 - \frac{k_1 - 1}{k_1} \gamma_1)], & (k_1 : \text{odd}),
\end{cases} \quad (A-4)
\]

\[
\beta_m''(\alpha_1, \alpha_2) = \begin{cases} 
\min\left\{ \frac{\beta}{2}, \frac{\beta}{2} (\alpha_2 - \gamma_1) \right\}, & (k_1 : \text{even}), \\
\min\left\{ \frac{\beta}{2}, \alpha_2 - \gamma_1 \right\}, & (k_1 : \text{odd}),
\end{cases} \quad (A-5)
\]

and Eqs. (A-4) and (A-5) are coincided with Eqs. (12) and (13).

(2) First $B_0'(x) = 2x$, $B_1'(x) = x$, $B_2'(x) = 1 + x^2$ and $B_3'(x) = 1$. By substituting Eq. (25) into $k_2 = 2$ and $\beta = 2\beta'$ and calculating Eq. (25) straightforwardly, we obtain $x_2 = \sqrt{\frac{\alpha_2 - \beta'}{R - \alpha_2 - \beta'}}$.

(3) By substituting (2) of Lemma 12 into the left hand side of Eq. (59) and calculating Eq. (59) straightforwardly, we obtain the right hand side of Eq. (59).