Average Coset Weight Distribution of Multi-Edge Type LDPC Code Ensembles

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SUMMARY  Multi-Edge type Low-Density Parity-Check codes (MET-LDPC codes) introduced by Richardson and Urbanke are generalized LDPC codes which can be seen as LDPC codes obtained by concatenating several standard (ir)regular LDPC codes. We prove in this paper that MET-LDPC code ensembles possess a certain symmetry with respect to their Average Coset Weight Distributions (ACWD). Using this symmetry, we drive ACWD of MET-LDPC code ensembles from ACWD of their constituent ensembles.

key words: average coset weight distribution, low-density parity-check codes, multi-edge type low-density parity-check codes, symmetry, weight distribution

1. Introduction

Low-Density Parity-Check (LDPC) codes invented by Gallager in 1963 [1] are linear error correcting codes with sparse parity-check matrices. Because of their excellent performance, LDPC codes have received much attention and many successful results have been established.

LDPC codes are well represented by the so-called Tanner graphs, i.e., bipartite graphs consisting of variable nodes and check nodes with sockets [2]. It is known that LDPC codes with structured bipartite graphs, e.g., Irregular Repeat Accumulate (IRA) codes [3] and Low-Density Generator Matrix (LDGM) codes [4], show excellent decoding performance. Multi-Edge type Low-Density Parity-Check codes (MET-LDPC codes) introduced by Richardson and Urbanke [5] give a standard framework which allows us to deal with these structured LDPC codes as a class.

MET-LDPC codes can be viewed as the codes obtained by concatenating several (ir)regular LDPC codes. An ensemble of MET-LDPC codes is defined by ensembles of constituent codes.

Codeword weight distributions play an important role in analyzing the performance of maximum likelihood decoding of the code [6]. Average codeword weight distributions of some LDPC codes ensembles, e.g., standard irregular LDPC codes [7], IRA codes [8], two-edge type LDPC codes [9], protograph LDPC codes [10] and combined LDPC codes [11], which are in the class of MET-LDPC codes, have been derived so far. However, the methods employed vary from paper to paper.

Wadayama derived Average Coset Weight Distribution (ACWD) of a regular LDPC code in [12], and then ACWD of a combined LDPC code ensemble in [11]. Since a coset weight distribution is a generalization of codeword weight distribution, we can derive a codeword weight distribution from a coset weight distribution. And the coset weight distribution is closely related to near-codewords [17] and trapping sets [16] for analysis of error floors.

In this paper, we prove that MET-LDPC code ensembles possess a certain symmetry with respect to their Average Coset Weight Distributions (ACWD). Using this symmetry and Wadayama’s method, we derive ACWD of MET-LDPC code ensembles from ACWD of their constituent ensembles.

2. Preliminaries

2.1 Cosets and Codewords

Let \(H\) be an \(m \times n\) parity-check matrix of a linear code. For a syndrome \(s \in \mathbb{F}_2^m\), we call a set of vectors \(C(s) := \{x \in \mathbb{F}_2^n \mid xH^T = s\}\) a coset corresponding to \(s\). We consider the coset weight distribution of a code, \(A_w(s) := \#\{x \in \mathbb{F}_2^n \mid x \in C(s), |x| = w\}\), where \(|x|\) denotes the Hamming weight of \(x\). Letting the syndrome \(s\) zero, we can get the weight distribution of codewords \(A_w(0)\).

2.2 Bipartite Graph Representation

LDPC codes are well represented by the so-called Tanner graphs [2]. Let \(G = (V, C, E)\) be a bipartite graph with a set \(V = \{v_1, v_2, \ldots, v_n\}\) of variable nodes, a set \(C = \{c_1, c_2, \ldots, c_m\}\) of check nodes and a set \(E\) of edges between check and variable nodes. A code defined by \(G\) has parity-check matrix \(H := (h_{ij})_{ij}\) such that \(h_{ij}\) is 1 if \(v_i\) and \(c_j\) are connected with an odd number of edges, and 0 otherwise.

In this paper, we use a graph \(G\) and a code defined by \(G\) interchangeably.
2.3 Block Partitioned Code

Assume that \( V \) and \( C \) have their partitions \( B \) and \( D \), respectively. That is, \( B \) is a set of subsets of \( V \) such that \( V = \bigcup_{b \in B} b \) and \( b_1 \cap b_2 = \emptyset \) for \( b_1 \neq b_2 \), \( b_1, b_2 \in B \). The same for \( D \). Elements of \( B \) and \( D \) are called variable blocks and check blocks, respectively. A graph \( G = (V, C, E) \) with partitions \( B \) and \( D \) of \( V \) and \( C \), respectively, is denoted by \( G = (V, C, E, B, D) \), and we call the corresponding code a block-partitioned code.

For example, a standard irregular LDPC code with degree distributions \((\lambda(x) = \sum_i \lambda_i x^{-i}, \rho(x) = \sum_i \rho_i x^{-i})\) and code length \( n \) can be viewed as a block-partitioned code with partitions \( B = \{b_1, b_2, \ldots\} \) and \( D = \{d_1, d_2, \ldots\} \), where \( b_i \) is the set of variable nodes of degree \( i \), \( d_j \) is the set of check nodes of degree \( j \), and the cardinalities of \( b_i \) and \( d_j \) are given by

\[
\# b_i = \frac{n\lambda_i}{\int_0^1 \lambda(x)dx}, \quad \# d_j = \frac{n\rho_j}{\int_0^1 \rho(x)dx},
\]

respectively.

In this paper, we exclusively consider an ensemble in which the graphs have the same sets of variable and check nodes and their partitions. That is, if \((V_1, C_1, E_1, B_1, D_1)\) and \((V_2, C_2, E_2, B_2, D_2)\) are elements of an ensemble, then \( V_1 = V_2, C_1 = C_2, B_1 = B_2 \) and \( D_1 = D_2 \).

2.4 Multi-Edge Type LDPC Code

An MET-LDPC graph is constructed from its constituent graphs by identifying nodes in some blocks in each constituent graph. We first show below an example to illustrate how an MET-LDPC code ensemble is constructed from its constituent code ensembles.

**Example 1:** We denote by \( S(n, \lambda(x), \rho(x)) \) a standard irregular LDPC code ensemble with degree distributions \((\lambda(x), \rho(x))\) and code length \( n \). Consider the following two ensembles of standard irregular LDPC codes:

\[
M^{(1)} = S \left( 5, \frac{2}{8}, \frac{6}{8}, \frac{2}{8}, \frac{3}{8} \right),
\]
\[
M^{(2)} = S \left( 4, \frac{2}{8}, \frac{6}{8}, \frac{2}{8}, \frac{3}{8} \right).
\]

We assume that \( M^{(i)} \) (\( i = 1, 2 \)) has variable blocks \( b_1^{(i)}, b_2^{(i)} \) and check blocks \( d_1^{(i)}, d_2^{(i)} \) with \#\( b_1^{(1)} = \#b_2^{(1)} \) and \#\( d_1^{(2)} = \#d_2^{(2)} \) as shown in Fig. 1. Then, the two ensembles \( M^{(1)} \) and \( M^{(2)} \) can be concatenated as \( M^{(1)} M^{(2)} \) by identifying the variable nodes in \( b_1^{(2)} \) with the variable nodes in \( b_2^{(2)} \), and the check nodes in \( d_1^{(2)} \) with the check nodes in \( d_2^{(2)} \). The resulting concatenated ensemble \( M^{(1)} M^{(2)} \) has partitions \( B = \{b_1^{(1)}, b_2^{(2)}\} \) for variable nodes and \( D = \{d_1^{(1)}, d_2^{(1)} = d_2^{(2)} \} \) for check nodes and two permutations of edges. It is noted that variable or check blocks which are identified can be in, general, more than one.
MET-LDPC code ensemble $M^{(1)}M^{(2)}\cdots M^{(t)}$ with $t$ constituent standard irregular LDPC code ensembles is equiprobable.

In [11], stacked ensembles with $B_1 = B_2$ and $D_1 \cap D_2 = \phi$ and concatenated ensembles with $D_1 = D_2$ and $B_1 \cap B_2 = \phi$ are only considered. The ensemble depicted in Fig. 1 is an example of the case which is not considered in [11].

2.5 Block-Partitioned Words and Syndromes

In this section, as simple extensions of usual (code)words and syndromes we define block-partitioned words and syndromes for a block-partitioned code $G = (V, C, E, B, D)$, in which we implicitly assume that $V, C, B$ and $D$ are ordered sets.

For $b \in B$ and $d \in D$ of a code $G = (V, C, E, B, D)$, let $x_b \in \mathbb{F}_2^{|c|}$ (resp. $s_d \in \mathbb{F}_2^{|d|}$) be a binary vector, each component of which corresponds to each variable node $v \in b$ (check node $c \in d$). Components in $x_b$ and in $s_d$ are denote by $x(v)$ and $s(c)$, respectively. Regarding $x_b$ ($b \in B$) and $s_d$ ($d \in D$) as component of a vector, we call $b$-tuple $x := (x_b)_{b \in B}$ and $d$-tuple $s := (s_d)_{d \in D}$ a block-partitioned word in $B$ and a block-partitioned syndrome in $D$. For example, if $B = \{b_1, b_2\}$ and $x_{b_1} = (1, 0)$ and $x_{b_2} = (0, 1, 1)$, then

$$x := (x_b)_{b \in B} = ((1, 0, 1), (1, 0, 1, 1)).$$

We also denote by $X_B$ the set of all block-partitioned words in $B$ be and by $S_D$ the set of all block-partitioned syndromes in $D$, respectively.

Moreover, we denote by $|x_b| \in \{0, 1, \ldots, \#b\}$ and $|s_d| \in \{0, 1, \ldots, \#d\}$ the (Hamming) weights of $x_b$ and $s_d$, respectively, and define $b$-tuple $|x| := (|x_b|)_{b \in B}$ and $d$-tuple $|s| := (|s_d|)_{d \in D}$, and call them the block-partitioned weight of word $x$ and syndrome $s$, respectively. We also denote by $W_B$ the set of all block-partitioned weights of words in $B$ and by $S_D$ the set of all block-partitioned weights of syndromes in $D$. For example, for $x := (x_b)_{b \in B}$ defined in Eq.(3), we have $|x| = (2, 3)$.

For a check node $c \in C$, denote by $V_c$ the set of variable nodes adjacent to $c$. We say that a word $x \in X_B$ has a syndrome $s \in S_D$ in the graph $G$, if

$$\sum_{v \in V_c} x(v) = s(c), \quad \text{for all } c \in C,$$

and denote this by $xH^T_G = s$.

3. ACWD of MET-LDPC Code Ensemble

In this section, we derive variable and check node block-partitioned ACWDs of MET-LDPC code ensembles. MET-LDPC codes are block-partitioned codes, therefore weight distributions of codewords and cosets are described by block-partitioned words and syndromes.

When we apply the coset weight distributions to block-partitioned codes, we naturally extend the weight to the block-partitioned weight. Instead of the number of codewords of weight $\ell$, we count the number of codewords with a given block-partitioned weight and block-partitioned syndrome weight, as explained below in detail.

In what follows, if there is no confusion we shall sometimes use terms by dropping block-partitioned.

3.1 Coset Weight Distribution

For a code $G = (V, C, E, B, D)$, we denote by $A^G(w, \sigma)$ the number of words $x \in X_B$ of weight $w \in W_B$ having syndromes $s \in S_D$ of weight $\sigma \in S_D$ and refer it to as coset weight distribution. That is:

$$A^G(w, \sigma) := \sum_{x:|w|=\sigma} \tilde{A}^G(x, s),$$

$$\tilde{A}^G(x, s) := \begin{cases} 1, & \text{if } xH^T_G = s \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

For a given MET-LDPC code ensemble $M$, the goal of this paper is to derive the average coset weight distribution $A^M(w, \sigma) := \mathbb{E}_{G \in M}[A^G(w, \sigma)]$. We use a notation $\mathbb{A}$ for coset weight distributions of a pair of vectors ($x, \sigma$) and a notation $A$ for coset weight distributions of a pair of weight ($w, \sigma$). We assume that every code in the constituent ensemble is equi-probable then $A^M(w, \sigma) = \sum_{G \in M} A^G(w, \sigma)$ since the probability assigned to every graph in $M$ is $\frac{1}{|M|}$.

Defining $\mathbb{A}^M(x, s) := \mathbb{E}_{G \in M}[\tilde{A}^G(x, s)]$, we easily obtain $\mathbb{A}^M(w, \sigma) = \sum_{x:|w|=\sigma} \mathbb{A}^M(x, s)$.

3.2 Symmetric Ensemble

Symmetry which will be defined in this section is a simple extention of the symmetry defined in [11]. We claim that MET-LDPC code ensembles possess the following fundamental property.

**Definition 1** (Symmetry): Let $M$ be a code ensemble with a partition $B$ for variable nodes and a partition $D$ for check nodes. We say that $M$ is symmetric if

$$\mathbb{A}^M(x^1, s^1) = \mathbb{A}^M(x^2, s^2),$$

for any two words $x^1, x^2 \in X_B$ and two syndromes $s^1, s^2 \in S_D$ with $|x^1| = |x^2|$ and $|s^1| = |s^2|$.

From Definition 1, $\mathbb{A}^M(x, s)$ depends only on $|x|$ and $|s|$. Then it is obvious that we can well define another distribution $B^M : W_B \times S_D \rightarrow \{0, 1, 2, \ldots\}$ by

$$B^M(|x|, |s|) := \mathbb{A}^M(x, s),$$

for a given symmetric ensemble $M$ with partitions $B$ and $D$. We also call $B^M(|x|, |s|)$ the block-partitioned ACWD of an ensemble $M$ (see Lemma 1 below).

\[\text{If there is no confusion, we may call a block-partitioned word and block-partitioned syndrome a word and syndrome, respectively.}\]
Example 2: We consider the block-partitioned ACWD of a standard irregular LDPC code ensemble \( S(n, \lambda(x), \rho(x)) \) and show that it is symmetric.

We denote by \( b_i \) the set of variable nodes of degree \( i \), and by \( d_j \) the set of check nodes of degree \( j \), and define the partitions on \( V \) and \( C \) by \( B := \{b_1, b_2, \ldots, \} \) and \( D := \{d_1, d_2, \ldots, \} \). Note that the cardinalities of \( b_i \) and \( d_j \) are given in Eq.(1). We also denote by \( \deg(b) \) the degree of variable nodes in a block \( b \in B \) and by \( \deg(d) \) the degree of check nodes in \( d \in D \).

Wadayama derived the ACWD of Gallager’s regular LDPC code ensembles, which are defined by parity check matrices, in [12] and the ACWD of regular LDPC code ensembles, which are defined by bipartite graphs, in [11]. His method can be easily extended to an irregular LDPC code ensemble, which are defined by bipartite graphs, in [11]. The number of possible configurations of edges is given by the same argument in the proof of [11, Lemma 3]. The number of edges of bipartite graph \( G \) is equal to \( \sum_{d \in D} \deg(d) \), and the number of inactive edge configuration is \( \#E - e \).

Therefore we have

\[
\bar{A}^M(x, s) = \frac{\text{coef} \left( \prod_{d \in D} (\alpha_d^+(u)^{\#d - |S_d|} \alpha_d^-(u)^{|S_d|}, u^e) \right) e!}{\#E!}.
\]

This equation obviously implies that a standard irregular LDPC code ensemble is symmetric.

The following lemma implies that we can derive \( A^M(w, \sigma) \) from \( B^M(w, \sigma) \) for a symmetric ensemble \( M \).

Lemma 1: For a symmetric ensemble \( M \) with partitions \( B \) and \( D \), it holds that

\[
A^M(w, \sigma) = \psi(M, w, \sigma) B^M(w, \sigma),
\]

where

\[
\psi(M, w, \sigma) := \prod_{b \in B} \left( \#b \right) \prod_{d \in D} \left( \#d \right).
\]

Proof: Recall the definition of symmetry in Definition 1. Then, from the definition of \( B^M \) and the observation that there are \( \prod_{b \in B} \left( \#b \right) \) words whose weight is \( w = (w_b)_{b \in B} \) and \( \prod_{d \in D} \left( \#d \right) \) syndromes whose weight is \( \sigma = (\sigma_d)_{d \in D} \), the claim of the lemma is obvious.

By Lemma 1, we see that it is sufficient to derive \( B^M(w, \sigma) \) in order to calculate \( A^M(w, \sigma) \).

3.3 Preservation of Symmetry

The following theorem tells us that a new concatenated ensemble preserves the symmetry of constituent ensembles.

Theorem 1 (Symmetry is Preserving): If two ensembles \( M^{(1)} \) and \( M^{(2)} \) are symmetric, their concatenated ensemble \( M^{(1)} M^{(2)} \) is also symmetric.

Proof: We use the same method as used in deriving ACWD of two-edge type LDPC code ensembles [9] and ACWD of concatenated matrix ensembles [11]. It is easily seen from the definitions of \( \bar{A}^{M^{(1)} M^{(2)}}(x, s) \) and Eq.(4) that \( \bar{A}^{M^{(1)} M^{(2)}}(x, s) \) can be derived by counting graphs \( G \) such that \( x H^T_G = s \). Any code which contributes \( \bar{A}^{M^{(1)} M^{(2)}}(x, s) \) are obtained by concatenating two graphs \( G^1 \) and \( G^2 \) which contribute \( \bar{A}^M(x^1, s^1) \) and \( \bar{A}^M(x^2, s^2) \), respectively, where \( x^1 \in S_{D^1} \) and \( x^2 \in S_{D^2} \) are any syndromes which satisfy (i) \( s^1_d + s^2_d = s_d \) for all \( d \in D_1 \cap D_2 \) (ii) \( s^1_d = s_d \) for all \( d \in D_1 \setminus D_2 \) (iii) \( s^2_d = s_d \) for all \( d \in D_2 \setminus D_1 \). Moreover \( x^1 \in X_{B_1} \) and \( x^2 \in X_{B_2} \), words defined by \( x^1_b := x_b \) for all \( b \in B_1 \), \( x^2_b := x_b \) for all \( b \in B_2 \), respectively. Therefore we have

\[
\bar{A}^{M^{(1)} M^{(2)}}(x, s) = \sum_{(S^1, S^2) \in S^{(1)} \times S^{(2)} \setminus S_D} \bar{A}^{M^{(1)}}(x^1, s^1) \bar{A}^{M^{(2)}}(x^2, s^2),
\]

where \( S^2(s) \) is defined by

\[
S^2(s) := \{(s^1, s^2) \in S_{D_1} \times S_{D_2} \mid s^1_d + s^2_d = s_d \text{ for all } d \in D_1 \cap D_2, \ s^1_d = s_d \text{ for all } d \in D_1 \setminus D_2, \ s^2_d = s_d \text{ for all } d \in D_2 \setminus D_1 \}.
\]
Moreover, for \( \mu^1, \mu^2 \in \Sigma_{D_1 \cap D_2} \), we define a set of syndrome pairs by

\[
S^2(\mu^1, \mu^2, s) := \{(s^1, s^2) \in S_{D_1} \times S_{D_2} | \|
\]

\[
\begin{align*}
& s^1_d + s^2_d = s \text{ for all } d \in D_1 \cap D_2, \\
& s^1_d = s_d \text{ for all } d \in D_1 \setminus D_2, \\
& s^2_d = s_d \text{ for all } d \in D_2 \setminus D_1, \\
& \mu^1_d = \#\{c \in d | s^1(c) = 1, s^2(c) = 0\}, \\
& \mu^2_d = \#\{c \in d | s^1(c) = s^2(c) = 1\},
\end{align*}
\]

for all \( d \in D_1 \cap D_2 \).

It can be easily seen that \( S^2(s) \) is partitioned by \( S^2(\mu^1, \mu^2, s) \) as follows.

\[
S^2(s) = \bigcup_{(\mu^1, \mu^2) \in M_{\Sigma}} S^2(\mu^1, \mu^2, s);
\]

\[
S^2(\mu^1, \mu^2, s) \cap S^2(v^1, v^2, s) = \phi,
\]

for \( (\mu^1, \mu^2) \neq (v^1, v^2) \),

where \( M_{\Sigma} \) is a set of syndrome pairs defined, for syndrome weight \( \sigma \in \Sigma_{D_1 \cup D_2} \), by

\[
M_{\Sigma} := \{(\mu^1, \mu^2) \in \Sigma_{D_1} \cap \Sigma_{D_2} \times \Sigma_{D_1 \cap D_2} | \mu^1_d \in \{0, 1, \ldots, \sigma_d\}, \mu^2_d \in \{0, 1, \ldots, \#d - \sigma_d\}, d \in D_1 \cap D_2\}.
\]

With this partition, Eq.(7) can be rewritten as follows.

\[
\tilde{A}^{M^1(1)M^2(2)}(x, s) = \sum_{(\mu^1, \mu^2) \in M_{\Sigma}} \sum_{(s^1, s^2) \in S^2(\mu^1, \mu^2, s)} \tilde{A}^{M^1(1)}(x^1, s^1) \tilde{A}^{M^2(2)}(x^2, s^2)
\]

\[
= \sum_{(\mu^1, \mu^2) \in M_{\Sigma}} \left( \sum_{d \in D_1 \cap D_2} \sum_{[s]} \left( \begin{array}{c}
\mu^1_d + \mu^2_d \\
|s|
\end{array} \right) \left( \begin{array}{c}
\#d - |s| \\
\mu^2_d
\end{array} \right) \right)
\]

\[
\times \tilde{A}^{M^1(1)}(x^1, t^1) \tilde{A}^{M^2(2)}(x^2, t^2),
\]

where \( t^1 \in S_{D_1}, t^2 \in S_{D_2} \) is any two syndromes such that

\[
|t^1_d| = \begin{cases}
\mu^1_d + \mu^2_d & \text{if } d \in D_1 \cap D_2, \\
|s| & \text{if } d \in D_1 \setminus D_2,
\end{cases}
\]

\[
|t^2_d| = \begin{cases}
|s| - \mu^1_d + \mu^2_d & \text{if } d \in D_1 \cap D_2, \\
|s| & \text{if } d \in D_2 \setminus D_1.
\end{cases}
\]

respectively. And (a) is due to the observation that

\[
\#S^2(s, \mu^1, \mu^2) = \prod_{d \in D_1 \cap D_2} \left( \begin{array}{c}
|s| \\
\mu^2_d
\end{array} \right) \left( \begin{array}{c}
\#d - |s| \\
\mu^2_d
\end{array} \right)
\]

and any \( (s^1, s^2) \in S^2(s, \mu^1, \mu^2) \) has weight \( |s^1| = |t^1| \) and \( |s^2| = |t^2| \) and that \( M^{(1)} \) and \( M^{(2)} \) are symmetric.

3.4 Average Coset Weight Distribution

**Theorem 2:** Let \( M^{(1)}, M^{(2)} \) be symmetric ensembles with partitions \((B_1, D_1)\) and \((B_2, D_2)\), respectively, yielding a concatenated ensemble \( M^{(1)}M^{(2)} \). For a given \( w = (w_b)_{b \in B_1 \cup B_2} \in W_{B_1 \cup B_2} \), define \( w^1 = (w_b)_{b \in B_1} \in W_{B_1} \) and \( w^2 = (w_b^2)_{b \in B_2} \in W_{B_2} \) by \( w^1_b := w_b \) for \( b \in B_1 \) and \( w^2_b := w_b \) for \( b \in B_2 \), respectively. Then, block-partitioned ACWD \( B^{M^{(1)}M^{(2)}}(w, \sigma) \) for \( \sigma = (\sigma_d)_{d \in D_1 \cup D_2} \in \Sigma_{D_1 \cup D_2} \) is represented by block-partitioned ACWDs \( B^{M^{(1)}}(w^1, \sigma^1) \) and \( B^{M^{(2)}}(w^2, \sigma^2) \) as follows.

\[
B^{M^{(1)}M^{(2)}}(w, \sigma) = \sum_{(\mu^1, \mu^2) \in M_{\Sigma}} \phi(\sigma, \mu^1, \mu^2)B^{M^{(1)}(w^1, \sigma^1)}(\mu^1)B^{M^{(2)}(w^2, \sigma^2)},
\]

where

\[
\phi(\sigma, \mu^1, \mu^2) := \prod_{d \in D_1 \cap D_2} \left( \begin{array}{c}
\sigma_d - \mu^1_d + \mu^2_d \\
\sigma_d
\end{array} \right).
\]

and two syndrome weights \( v^1 \in \Sigma_{D_1} \), \( v^2 \in \Sigma_{D_2} \) are defined by

\[
v^1_d := \begin{cases}
\mu^1_d + \mu^2_d & \text{if } d \in D_1 \cap D_2, \\
\sigma_d & \text{if } d \in D_1 \setminus D_2,
\end{cases}
\]

\[
v^2_d := \begin{cases}
\sigma_d - \mu^1_d + \mu^2_d & \text{if } d \in D_1 \cap D_2, \\
\sigma_d & \text{if } d \in D_2 \setminus D_1.
\end{cases}
\]

respectively.

**Proof:** Obvious from Eq.(8) in the proof of Theorem 1 and Lemma 1. \( \square \)

A general MET-LDPC code ensemble \( M \) is obtained by concatenating \( t \) standard (ir)regular LDPC code ensembles \( M^{(1)}, M^{(2)}, \ldots, M^{(t)} \) as in Eq.(2). Note that

1. as shown in Example 2, all standard (ir)regular LDPC code ensembles are symmetric and their ACWDs can be calculated by Eq.(6)
2. by Theorem 1, a concatenated MET-LDPC code ensemble obtained from two symmetric ensembles is also symmetric
3. by Theorem 2, we can calculate the ACWD of an MET-LDPC code ensemble of two symmetric constituent code ensembles from their ACWDs
4. Eq.(2) means that if we can derive ACWD of a code ensemble obtained by concatenating two constituent ensembles, we can derive ACWD of any MET-LDPC code ensemble.

Therefore, we can conclude that block-partitioned ACWD of any MET-LDPC code ensemble can be derived from block-partitioned ACWDs of constituent standard (ir)regular LDPC code ensembles.

Finally, it is noted that the conventional average (non-block-partitioned) coset weight distribution \( A_d(s) \), i.e., the
average number of words of weight $\ell$ in a coset corresponding to syndrome $s \in S_{\ell}$, is calculated as

$$A_{\ell}(s) = \frac{1}{\prod_{d \in D} \binom{|A|}{w}} \sum_{w_{i}=\sum_{d \in D} w_{d}} A^{\ell}(w_{s}).$$

The conventional average (non-block-partitioned) codeword weight distribution $A_{\ell}$, i.e., the average number of codewords of weight $\ell$, is readily given by $A_{\ell} = A_{\ell}(\emptyset)$.

4. Conclusions and Discussions

By introducing the concept of symmetry on block-partitioned ACWD in MET-LDPC code ensembles and their constituent code ensembles, we have succeeded in deriving the block-partitioned ACWD of an MET-LDPC code ensemble from block-partitioned ACWD of its constituent code ensembles.

In [1], [6], by upper bounding the weight distributions of LDPC codes, upper bounds of error probabilities under maximum likelihood decoding is derived. While the computation of coset weight distributions for interesting code ensembles remains intractable, we believe that we have made a promising step or two.

It is known that stopping sets play an important role on the performance of an iterative decoding over the erasure channels [14]. Orlitsky et al. derived the stopping set distribution for standard irregular LDPC code ensembles in [15] and Ikegaya et al. derived the one for two-edge type LDPC code ensembles in [9], which are the simplest examples of MET-LDPC code ensembles. We are expecting that average stopping set distribution for MET-LDPC code ensembles can be derived by introducing an appropriate symmetry on certain generalized coset weight distribution.

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References

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